TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 273, Number 2, October 1982

REPRODUCING KERNELS AND BILINEAR SUMS FOR q-RACAH AND q-WILSON POLYNOMIALS

BY
MIZAN RAHMAN[1]

ABSTRACT. A five-parameter family of reproducing kernels is constructed for q-Racah polynomials. Special cases for q-Hahn and little q-Jacobi polynomials are considered by selecting the parameters appropriately. Corresponding bilinear sums are also obtained for a whole range of q-orthogonal polynomials. As a special case, some product formulas are obtained for q-Racah and q-Wilson polynomials.

1. Introduction. In [9] we constructed a five-parameter family of reproducing kernels for Racah polynomials defined by a balanced ${}_4F_3$ series:

$$W_n(x) = {}_4F_3\left[\begin{matrix} -n, & n+\alpha+\beta+1, & -x, & x+\gamma-N \\ & \alpha+1, & -N, & \beta+\gamma+1 \end{matrix}\right], \tag{1.1}$$

where $-N$ is the largest negative integer that appears in the denominator and $x, n = 0, 1, \ldots, N$.

The purpose of this paper is to seek q-analogues of the main results of [9] and to discuss some interesting special cases. With the recent discovery of a q-analogue of (1.1) by Askey and Wilson [4] it is natural to expect that most, if not all, known results of Racah polynomials ought to have q-extensions. This remark seems especially true in view of the transformation properties of these basic polynomials which are defined by

$$\begin{aligned} W_n(x; q) &\equiv W_n(x; a, b, c, N; q) \\ &= {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n+1}ab, & q^{-x}, & cq^{x-N} \\ & aq, & bcq, & q^{-N} \end{matrix}; q, q\right] \end{aligned} \tag{1.2}$$

and are called q-Racah polynomials, where x, n and N have the same meaning as in (1.1), and, a, b, c are complex parameters that are restricted only by the requirement that the denominator parameters aq and bcq do not lead to a zero factor before any in the numerator does. The ${}_4\phi_3$ series is balanced, which means that the product of

Received by the editors November 19, 1980 and, in revised form, April 23, 1981.
AMS (MOS) subject classifications (1970). Primary 33A65, 33A70; Secondary 33A30.
Key words and phrases. q-Racah polynomials, q-Wilson polynomials, q-Hahn polynomials, connection relation, bilinear formula, reproducing kernel, very well-poised basic hypergeometric series, balanced basic hypergeometric series, continuous q-Jacobi polynomials, Watson and Bateman-type product formulas.
[1] This work was supported by Natural Sciences and Engineering Research Council (Canada) under grant #A6197. The contents of this paper were presented at the Annual Meeting of the Canadian Mathematical Society at Victoria, B. C., December 10–12, 1981.

0002-9947/82/0000-0873/$7.50

the denominator parameters is q times that of the numerator parameters, and is a special type of a basic hypergeometric series defined by

$$(1.3)\qquad {}_{k+1}\phi_k\left[\begin{matrix} a_1, a_2, \ldots, a_{k+1} \\ b_1, b_2, \ldots, b_k \end{matrix}; q, x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{k+1})_n}{(q)_n (b_1)_n \cdots (b_k)_n} x^n,$$

where

$$(1.4)\qquad (a)_n \equiv (a; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-a)(1-qa)\cdots(1-aq^{n-1}), & n = 1, 2, \ldots. \end{cases}$$

The transformation property we just alluded to is a q-analogue of Whipple's formula for the transformation of a balanced ${}_4F_3$ of type (1.1) and is given by

$$(1.5)\qquad \begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n+1}ab, & q^{-x}, & cq^{x-N} \\ & aq, & bcq, & q^{-N} \end{matrix}; q, q\right] \\ &= \frac{(bq)_n (ac^{-1}q)_n}{(aq)_n (bcq)_n} c^n \, {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n+1}ab, & q^{x-N}, & c^{-1}q^{-x} \\ & bq, & ac^{-1}q, & q^{-N} \end{matrix}; q, q\right]. \end{aligned}$$

Using (1.2), (1.5) and the well-known sum of a very well-poised ${}_6\phi_5$ **[11]**,

$$(1.6)\qquad \begin{aligned} &{}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq^{n+1} \end{matrix}; q, \frac{aq^{n+1}}{bc}\right] \\ &\qquad = \frac{(aq)_n (aq/bc)_n}{(aq/b)_n (aq/c)_n}, \end{aligned}$$

one can show that the polynomials $W_n(x; q)$ satisfy an orthogonality relation **[4]**

$$(1.7)\qquad \sum_{x=0}^{N} \rho(x) W_m(x; q) W_n(x; q) = h_n \delta_{m,n}$$

where the weight function $\rho(x)$ is defined by

$$(1.8)\qquad \rho(x) = \frac{(cq^{-N})_x \left(q\sqrt{cq^{-N}}\right)_x \left(-q\sqrt{cq^{-N}}\right)_x (aq)_x (bcq)_x (q^{-N})_x}{(q)_x \left(\sqrt{cq^{-N}}\right)_x \left(-\sqrt{cq^{-N}}\right)_x (ca^{-1}q^{-N})_x (b^{-1}q^{-N})_x (cq)_x} (abq)^{-x}$$

and the normalization constant h_n by

$$(1.9)\qquad h_n = h_0 \frac{(q)_n \left(\sqrt{abq}\right)_n \left(-\sqrt{abq}\right)_n (bq)_n (ac^{-1}q)_n (abq^{N+2})_n (cq^{-N})^n}{(abq)_n \left(q\sqrt{abq}\right)_n \left(-q\sqrt{abq}\right)_n (aq)_n (bcq)_n (q^{-N})_n},$$

with

$$(1.10)\qquad h_0 = \frac{(cq^{1-N})_N (a^{-1}b^{-1}q^{-N-1})_N}{(ca^{-1}q^{-N})_N (b^{-1}q^{-N})_N} = \frac{(c^{-1})_N (abq^2)_N}{(bq)_N (ac^{-1}q)_N}.$$

The formula for the total weight h_0 follows from (1.6) and the use of the identity **[11**, p. 241]

$$(1.11)\qquad (aq^{-n})_n = (-a)^n q^{-n(n+1)/2} (q/a)_n.$$

The duality of $W_n(x; q)$ in x and n is obvious in definition (1.2) and one can exploit that to write the dual orthogonality relation

$$\sum_{n=0}^{N} \left(h_0 h_n^{-1}\right) W_n(x; q) W_n(y; q) = \frac{h_0 \delta_{x,y}}{\rho(x)}. \tag{1.12}$$

A limiting case of the q-Racah polynomials, discovered by Hahn in 1949 [7], corresponds to setting $c = 0$ in (1.2):

$$Q_n(x; q) = {}_3\Phi_2\left[\begin{matrix} q^{-n}, & q^{n+1}ab, & q^{-x} \\ & aq, & q^{-N} \end{matrix}; q, q\right]. \tag{1.13}$$

There are transformation formulas for $Q_0(x; q)$. But the most important one can be worked out from (1.5) by taking the limit $c \to 0$. Thus one has

(1.14)

$$Q_n(x; q) = (-1)^n \frac{(bq)_n}{(aq)_n} a^n q^{n(n+1)/2} {}_3\Phi_2\left[\begin{matrix} q^{-n}, & q^{n+1}ab, & q^{x-N} \\ & bq, & q^{-N} \end{matrix}; q, q^{-x}/a\right].$$

Andrews and Askey studied these polynomials and found a weight function for them a few years ago [3]. Another limiting case that will be of interest to us is known in the current literature as little q-Jacobi polynomials and is obtained from (1.13) by taking the limits $x, N \to \infty$ such that $N - x = r$ where r takes on nonnegative integer values. These polynomials were also studied by Hahn [6], as well as Andrews and Askey [2, 3], and are defined by

$$p_n(x; \alpha, \beta \mid q) = {}_2\Phi_1\left[\begin{matrix} q^{-n}, & q^{n+1}\alpha\beta \\ & \alpha q \end{matrix}; q, qx\right] \tag{1.15}$$

where $x = q^r$, $r = 0, 1, \ldots$.

In [1] Al-Salam and Ismail found a reproducing kernel for $p_n(x; \alpha, \beta \mid q)$ and a corresponding bilinear formula which will be seen in this paper to follow as a very special case of a rather general family of reproducing kernels for the q-Racah polynomials $W_n(x; q)$. We first take five arbitrary parameters a_1, a_2, b_1, b_2, b_3 and define two different q-Racah polynomials:

$$W_n^{(1)}(x; q) = {}_4\Phi_3\left[\begin{matrix} q^{-n}, & q^{n+1}a_1a_2b_2b_3, & q^{-x}, & b_1b_2^{-1}cq^{x-N} \\ & a_1b_1q, & q^{-N}, & a_2b_3cq \end{matrix}; q, q\right] \tag{1.16}$$

and

$$W_n^{(2)}(x; q) = {}_4\Phi_3\left[\begin{matrix} q^{-n}, & q^{n+1}a_1a_2b_2b_3, & q^{-x}, & cq^{x-N} \\ & a_1b_2q, & q^{-N}, & a_2b_3cq \end{matrix}; q, q\right]. \tag{1.17}$$

Obviously $W_n^{(1)}(x; q) = W_n^{(2)}(x; q)$ when $b_1 = b_2$.

We then show that they are related by a connection relation

$$\sum_{y=0}^{N} K_N(x, y) W_n^{(2)}(y; q) = \lambda_n W_n^{(1)}(x; q) \tag{1.18}$$

where the "eigenvalue" λ_n is given by a balanced ${}_4\phi_3$:

$$\lambda_n = {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n+1}a_1a_2b_2b_3, & b_2, & b_1^{-1}b_2b_3 \\ & a_1b_2q, & b_2b_3, & a_2b_1^{-1}b_2b_3q \end{matrix}; q, q\right] \tag{1.19}$$

and the kernel $K_N(x, y)$ has the double sum form:

$$\begin{aligned} K_N(x, y) = A_N(x, y) &\sum_{i=0}^{x\wedge y} \frac{\left(cb_2^{-1}q^{-N}\right)_i\left(q\sqrt{cb_2^{-1}q^{-N}}\right)_i\left(-q\sqrt{cb_2^{-1}q^{-N}}\right)_i(a_1q)_i}{(q)_i\left(\sqrt{cb_2^{-1}q^{-N}}\right)_i\left(-\sqrt{cb_2^{-1}q^{-N}}\right)_i\left(ca_1^{-1}b_2^{-1}q^{-N}\right)_i} \\ &\cdot \frac{\left(b_1b_2^{-1}cq^{x-N}\right)_i\left(b_2^{-1}b_3^{-1}q^{1-N}\right)_i\left(cb_2^{-1}q\right)_i\left(cq^{y-N}\right)_i(q^{-x})_i(q^{-y})_i}{\left(b_1^{-1}q^{1-x}\right)_i(b_3c)_i(q^{-N})_i\left(b_2^{-1}q^{1-y}\right)_i\left(cb_2^{-1}q^{1+x-N}\right)_i\left(cb_2^{-1}q^{1+y-N}\right)_i}\left(\frac{b_3}{a_1b_1}\right)^i \\ &\cdot {}_{10}\phi_9\left[\begin{matrix} b_3^{-1}c^{-1}q^{-N}, & q\sqrt{b_3^{-1}c^{-1}q^{-N}}, & -q\sqrt{b_3^{-1}c^{-1}q^{-N}}, & a_2q, & b_1^{-1}b_2c^{-1}q^{-x}, \\ & \sqrt{b_3^{-1}c^{-1}q^{-N}}, & -\sqrt{b_3^{-1}c^{-1}q^{-N}}, & a_2^{-1}b_3^{-1}c^{-1}q^{-N}, & b_1b_2^{-1}b_3^{-1}q^{1+x-N}, \end{matrix}\right. \\ &\qquad \left.\begin{matrix} c^{-1}q^{-y}, & b_2^{-1}b_3^{-1}q^{1+i-N}, & b_3^{-1}c^{-1}q^{1-i}, & q^{x-N}, & q^{y-N} \\ b_3^{-1}q^{1+y-N}, & b_2c^{-1}q^{-i}, & q^{i-N}, & b_3^{-1}c^{-1}q^{1-x}, & b_3^{-1}c^{-1}q^{1-y} \end{matrix}; q, \frac{b_1}{a_2b_3}\right], \end{aligned} \tag{1.20}$$

where

$$\begin{aligned} A_N(x, y) = &\frac{(q)_N(a_2b_3cq)_N\left(a_1b_2c^{-1}q\right)_N}{(b_2b_3)_N(b_3c)_N\left(b_2c^{-1}\right)_N} \cdot \frac{(b_1)_x(b_3c)_x(b_2)_y(b_3c)_y}{(a_1b_1q)_x(a_2b_3cq)_x(q)_y(cq)_y} \\ &\cdot \frac{\left(b_2c^{-1}\right)_{N-x}\left(b_2b_3b_1^{-1}\right)_{N-x}(b_3)_{N-y}\left(b_2c^{-1}\right)_{N-y}}{\left(a_1b_2c^{-1}q\right)_{N-x}\left(a_2b_2b_3b_1^{-1}q\right)_{N-x}(q)_{N-y}\left(c^{-1}\right)_{N-y}} \cdot \frac{1 - cq^{2y-N}}{1 - cq^{y-N}}, \end{aligned} \tag{1.21}$$

and $x \wedge y = \min(x, y)$.

Note that the ${}_{10}\phi_9$ series in (1.20) is terminating and very well-poised, and so is the second series implied in it. However, transformations of these series are, in general, not possible unless they have the additional property of being balanced, that is, the product of the denominator parameters is q times that of the numerator parameters. For the ${}_{10}\phi_9$ shown explicitly in (1.20) this would require $a_2q = b_1/b_3$, while for the second series one would need $a_1q = b_3/b_1$.

One may also note the kernel $K_N(x, y)$ is positive under the conditions:

$$0 < a_1q, a_2q, b_1, b_2, b_3, b_2b_3/b_1 < 1, \qquad ca_i^{-1}b_j^{-1}q^{-N} < 1, \tag{1.22}$$

or,

$$a_1q^{1+N}, a_2q^{1+N}, b_1q^N, b_2q^N, b_3q^N, a_2b_2b_3b_1^{-1}q^{N+1} > 1, \qquad ca_ib_jq < 1, \tag{1.23}$$

where $0 < q < 1$ and $i = 1, 2; j = 1, 2, 3$.

Using (1.5) and (1.19) one finds that

$$\lambda_n = \frac{(a_1^{-1}q^{-n})_n (b_3^{-1}q^{1-n})_n}{(a_1b_2q)_n (b_2b_3)_n} (a_1b_2b_3q^n)^n$$

$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, & b_2, & a_2q, & a_1^{-1}b_1^{-1}q^{-n} \\ & a_2b_1^{-1}b_2b_3q, & a_1^{-1}q^{-n}, & b_3^{-1}q^{1-n} \end{matrix}; q, q\right]$$

$$(1.24) \qquad = \frac{(q)_n (a_1^{-1}b_1^{-1}q^{-n})_n}{(a_1b_2q)_n (b_2b_3)_n} (a_1b_2b_3q^n)^n$$

$$\cdot \sum_{k=0}^{n} \frac{(q^{-n})_k (b_2)_k (a_2q)_k (a_1^{-1}q^{k-n})_{n-k} (b_3^{-1}q^{1+k-n})_{n-k}}{(q)_n (q)_k (a_1^{-1}b_1^{-1}q^{k-n})_{n-k} (a_2b_1^{-1}b_2b_3q)_k} q^k$$

$$= \frac{(q)_n (a_1b_1q)_n}{(a_1b_2q)_n (b_2b_3)_n} b_2^n q^{n(n+1)/2}$$

$$\cdot \sum_{k=0}^{n} \frac{(a_1q)_{n-k} (b_2)_k (b_3)_{n-k} (a_2q)_k}{(q)_k (q)_{n-k} (a_1b_2q)_{n-k} (a_2b_1^{-1}b_2b_3q)_k} \left(\frac{b_3}{b_1q}\right)^k,$$

by (1.11).

It is interesting to note that λ_n is independent of c and N and is positive under conditions (1.22) and (1.23).

In the next section we prove the connection relation (1.18). In §3 we discuss some special cases and in §4 we obtain some bilinear formulas. In §5 we deduce some Bateman-type and Watson-type product formulas for q-Racah and q-Wilson polynomials in general, and, for the continuous q-Jacobi polynomials, in particular.

2. Construction of the kernel $K_N(x, y)$. Applying (1.5) to the balanced series in (1.17) we obtain

$$(2.1) \qquad W_n^{(2)}(y; q) = \frac{(a_2b_3q)_n (a_1a_2b_2b_3q^{N+2})_n}{(a_1b_2q)_n (q^{-N})_n} (a_2b_3q^{N+1})^{-n} W_n'(y; q),$$

where

$$(2.2) \quad W_n'(y; q) = \sum_{r=0}^{n} \frac{(q^{-n})_r (q^{n+1}a_1a_2b_2b_3)_r (a_2b_3cq^{y+1})_r (a_2b_3q^{N-y+1})_r}{(q)_r (a_2b_3q)_r (a_2b_3cq)_r (a_1a_2b_2b_3q^{N+2})_r} q^r.$$

Using the q-analogue of the Pfaff-Saalschutz theorem [**11**, equation (IV, 4), p. 247], namely,

$$(2.3) \qquad {}_3\phi_2\left[\begin{matrix} q^{-k}, & a, & b \\ & c, & abc^{-1}q^{1-k} \end{matrix}; q, q\right] = \frac{(c/a)_k (c/b)_k}{(c)_k (c/ab)_k},$$

we obtain

$$(a_2b_3cq^{1+y})_r (a_2b_3q^{N-y+1})_r = (a_2b_3^2cq^{1+j})_r$$

$$(2.4) \qquad \cdot \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} \frac{(b_3q^{j-y})_k (b_3cq^{y-N+j})_k}{(a_2b_3^2cq^{1+j})_k} (a_2q^{N-j+1})_{r-k} (a_2q^{N+1-j})^k,$$

where j is a nonnegative integer such that $y \leqslant j \leqslant N$ and $\left[{r \atop k}\right]$ is the q-binomial coefficient defined by $\left[{r \atop k}\right] = (q)_r/(q)_k(q)_{r-k}$. In deriving the r.h.s. of (2.4) from (2.3) we need to make use of the following identities [**11**, p. 241].

$$(2.5) \qquad (a)_{N-n} = \frac{(a)_N q^{n(n+1)/2-Nn}}{(a^{-1}q^{1-N})_n(-a)^n},$$

$$(2.6) \qquad (aq^{-n})_N = \frac{(a)_N(q/a)_n}{(a^{-1}q^{1-N})_n} q^{-Nn}.$$

Let us now compute the sum

$$(2.7) \qquad \begin{aligned} L_1(i, j) = \sum_{y=i}^{j} b_2^{-y} \frac{1-cq^{2y-N}}{1-cq^{2i-N}} \frac{(cq^{i-N})_y(b_3cq^{j-N})_y}{(cq^{1+j-N})_y(b_2^{-1}cq^{1+i-N})_y} \\ \cdot \frac{(q^{y-i+1})_\infty(q^{j-y+1})_\infty}{(b_2q^{y-i})_\infty(b_3q^{j-y})_\infty} W_n'(y; q). \end{aligned}$$

One may feel somewhat mystified by the appearance of the coefficients on the right-hand side, but this is the obvious q-analogue of a similar object we considered in [**9**] and is motivated simply by the requirement that these coefficients together with the y-dependent terms in (2.4) will be summable by a very well-poised ${}_6\phi_5$ sum (1.6).

Through a somewhat tedious but straightforward calculation one can, in fact, show that the use of (2.4), the identities (2.5), (2.6), and the summation formula (1.6) yield the following result:

$$(2.8) \qquad \begin{aligned} L_1(i, j) = C(i, j) \sum_{r=0}^{n} \frac{(q^{-n})_r(q^{n+1}a_1a_2b_2b_3)_r(a_2b_3^2cq^{1+j})_r(a_2q^{N-j+1})_r}{(q)_r(a_2b_3q)_r(a_2b_3cq)_r(a_1a_2b_2b_3q^{N+2})_r} q^r \\ \cdot {}_4\phi_3\left[\begin{matrix} q^{-r}, & b_3 & b_3cq^{i+j-N}, & b_2b_3q^{j-i} \\ & a_2^{-1}q^{j-N-r}, & b_2b_3, & a_2b_3^2cq^{1+j} \end{matrix}; q, q\right] \end{aligned}$$

where

$$(2.9) \quad C(i, j) = \frac{(q)_\infty^2}{(b_2)_\infty(b_3)_\infty} b_2^{-j} \frac{(b_2b_3)_{j-i}(b_3cq^{j-N})_i(cq^{1-N})_j}{(q)_{j-i}(cq^{1-N})_i(b_2^{-1}cq^{i+1-N})_j} \cdot \frac{1-cq^{i-N}}{1-cq^{2i-N}}.$$

One fortunate, though expected, feature of (2.8) is that the ${}_4\phi_3$ series on the r.h.s. is balanced, hence we may apply the transformation (1.5) on it as often as necessary. The transformation we require specifically is the following:

$$(2.10) \qquad \begin{aligned} s_4\phi_3\left[\begin{matrix} q^{-r}, & b_3, & b_3cq^{i+j-N}, & b_2b_3q^{j-i} \\ & a_2^{-1}q^{j-N-r}, & b_2b_3, & a_2b_3^2cq^{1+j} \end{matrix}; q, q\right] \\ = \frac{(b_2)_r(a_2b_3cq^{1+j})_r}{(b_2b_3)_r(a_2b_3^2cq^{1+j})_r} b_3^r \\ \cdot {}_4\phi_3\left[\begin{matrix} q^{-r}, & b_3, & a_2^{-1}b_3^{-1}c^{-1}q^{-i-r}, & a_2^{-1}b_2^{-1}b_3^{-1}q^{i-N-r} \\ & b_2^{-1}q^{1-r}, & a_2^{-1}b_3^{-1}c^{-1}q^{-j-r}, & a_2^{-1}q^{j-N-r} \end{matrix}; q, q\right]. \end{aligned}$$

It may be mentioned that the coefficient of ${}_4\phi_3$ on the r.h.s. does not follow directly from (1.5), but only after using (1.11). Now, by (2.5),

(2.11)

$$\frac{(b_2)_r(a_2b_3cq^{1+j})_r(a_2q^{N-j+1})_r}{(b_2^{-1}q^{1-r})_k(a_2^{-1}b_3^{-1}c^{-1}q^{-j-r})_k(a_2^{-1}q^{j-N-r})_k}$$
$$= (b_2)_{r-k}(a_2b_3cq^{1+j})_{r-k}(a_2q^{N-j+1})_{r-k}(-a_2^2b_2b_3cq^{N+2})^k q^{3rk-3k(k+1)/2}.$$

Using this in (2.10) and then in (2.8) we obtain

$$L_1(i,j) = C(i,j)\sum_{r=0}^{n}\frac{(q^{-n})_r(q^{n+1}a_1a_2b_2b_3)_r(b_3q)^r}{(q)_r(a_2b_3q)_r(a_2b_3cq)_r(a_1a_2b_2b_3q^{N+2})_r(b_2b_3)_r}$$

(2.12)
$$\cdot\sum_{k=0}^{r}\frac{(q^{-r})_k(b_3)_k(a_2^{-1}b_3c^{-1}q^{-i-r})_k(a_2^{-1}b_2^{-1}b_3^{-1}q^{i-N-r})_k}{(q)_k}(-a_2^2b_2b_3cq^{N+3})^k$$
$$\cdot q^{3rk-3k(k+1)/2}\cdot(b_2)_{r-k}(a_2b_3cq^{1+j})_{r-k}(a_2q^{N-j+1})_{r-k}.$$

Next we compute the following sum:

$$L_2(i) = \sum_{j=x}^{N}\frac{1-cb_3q^{2j-N}}{1-cb_3q^{-N}}\frac{(a_2b_3cq)_j(b_3cq^{x-N})_j}{(b_3cq)_j(b_1b_2^{-1}cq^{1+x-N})_j}$$

(2.13)
$$\cdot\frac{(q^{j-x+1})_\infty(q^{N-j+1})_\infty}{(b_1^{-1}b_2b_3q^{j-x})_\infty(a_2q^{N-j+1})_\infty}(b_1b_2^{-1}b_3^{-1})^j$$
$$\cdot C^{-1}(i,j)L_1(i,j).$$

Again, the structure of the r.h.s. is aimed at the summability over j by using (1.6). Substituting (2.12) and (2.13), simplifying, summing, and again simplifying by means of (1.11), (2.5) and (2.6), we obtain the following form:

$$L_2(i) = A(x)\sum_{r=0}^{n}\frac{(q^{-n})_r(q^{n+1}a_1a_2b_2b_3)_r(b_3q)^r}{(q)_r(a_2b_3q)_r(a_2b_3cq)_r(a_1a_2b_2b_3q^{N+2})_r(b_2b_3)_r}$$

(2.14)
$$\cdot\sum_{k=0}^{r}\frac{(q^{-r})_k(b_3)_k(a_2q)_{r-k}(a_2b_3cq^{1+x})_{r-k}(a_2b_2b_3b_1^{-1}q^{N-x+1})_{r-k}(b_2)_{r-k}}{(q)_k(a_2b_2b_3b_1^{-1}q)_{r-k}}$$
$$\cdot(-a_2^2b_2b_3cq^{N+3})^k q^{3rk-3k(k+1)/2}(a_2^{-1}b_3^{-1}c^{-1}q^{-r-i})_k(a_2^{-1}b_2^{-1}b_3^{-1}q^{i-N-r})_k,$$

where

$$(2.15)\quad \begin{aligned} A(x) = {} & \frac{(q)_\infty^2}{(a_2q)_\infty(b_1^{-1}b_2b_3)_\infty} \frac{(b_3^{-1}c^{-1})_N(a_2b_2b_3b_1^{-1}q)_N}{(q)_N(b_2b_1^{-1}c^{-1})_N} \\ & \cdot \frac{(a_2b_3cq)_x(b_2^{-1}b_1cq^{1-N})_x(q^{-N})_x}{(b_3cq^{-N})_x(b_1b_2^{-1}cq)_x(b_1a_2^{-1}b_2^{-1}b_3^{-1}q^{-N})_x}\left(\frac{b_1}{a_2b_2b_3}\right)^x. \end{aligned}$$

Finally we compute the sum

$$(2.16)\quad \begin{aligned} L(x) = {} & \sum_{i=0}^{x} \frac{(cb_2^{-1}q^{-N})_i\left(q\sqrt{cb_2^{-1}q^{-N}}\right)_i\left(-q\sqrt{cb_2^{-1}q^{-N}}\right)_i(b_1b_2^{-1}cq^{x-N})_i}{\left(\sqrt{cb_2^{-1}q^{-N}}\right)_i\left(-\sqrt{cb_2^{-1}q^{-N}}\right)_i(cb_2^{-1}q^{x+1-N})_i(ca_1^{-1}b_2^{-1}q^{-N})_i} \\ & \cdot \frac{(q^{i+1})_\infty(q^{x-i+1})_\infty}{(a_1q^{i+1})_\infty(b_1q^{x-i})_\infty}(qa_1)^{-i}\cdot A^{-1}(x)L_2(i). \end{aligned}$$

Substitution of (2.14) in (2.16) now leads to the following form:

$$(2.17)\quad \begin{aligned} L(x) = {} & \frac{(q)_\infty^2}{(b_1)_\infty(a_1q)_\infty}\frac{(b_1)_x}{(q)_x}\sum_{r=0}^{n}\frac{(q^{-n})_r(q^{n+1}a_1a_2b_2b_3)_r(b_3q)^r}{(q)_r(a_2b_3q)_r(a_2b_3cq)_r(b_2b_3)_r(a_1a_2b_2b_3q^{N+2})_r} \\ & \cdot\sum_{k=0}^{r}\frac{(q^{-r})_k(b_3)_k(b_2)_{r-k}(a_2q)_{r-k}(a_2b_3cq^{1+x})_{r-k}(a_2b_2b_3b_1^{-1}q^{N-x+1})_{r-k}}{(q)_k(a_2b_2b_3b_1^{-1}cq)_{r-k}} \\ & \cdot\left(-a_2^2b_2b_3cq^{N+3}\right)^k\cdot q^{3rk-3k(k+1)/2}\xi_{k,r}, \end{aligned}$$

where

(2.18)

$$\begin{aligned} \xi_{k,r} = {} & (a_2^{-1}b_3^{-1}c^{-1}q^{-r})_k(a_2^{-1}b_2^{-1}b_3^{-1}q^{-N-r})_k \\ & \cdot {}_8\phi_7\left[\begin{matrix} cb_2^{-1}q^{-N}, & q\sqrt{cb_2^{-1}q^{-N}}, & -q\sqrt{cb_2^{-1}q^{-N}}, & a_1q, \\ & \sqrt{cb_2^{-1}q^{-N}}, & \sqrt{cb_2^{-1}q^{-N}}, & ca_1^{-1}b_2^{-1}q^{-N}, \end{matrix}\right. \\ & \left.\begin{matrix} b_1b_2^{-1}cq^{x-N}, & a_2b_3cq^{1+r}, & a_2^{-1}b_2^{-1}b_3^{-1}q^{k-N-r}, & q^{-x} \\ b_1^{-1}q^{1-x}, & a_2^{-1}b_2^{-1}b_3^{-1}q^{-N-r}, & a_2b_3cq^{1+r-k}, & cb_2^{-1}q^{1+x-N} \end{matrix}; q, a_1^{-1}b_1^{-1}q^{-k}\right]. \end{aligned}$$

However, the ${}_8\phi_7$ series on the right is very well-poised and has the structure that enables us to transform it to a balanced ${}_4\phi_3$ by means of the formula [**11**, equation (3.4.1.5), p. 100]

$$(2.19)\quad \begin{aligned} & {}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix}; q, \frac{a^2q^{n+2}}{bcde}\right] \\ & = \frac{(aq)_n(aq/de)_n}{(aq/d)_n(aq/e)_n}{}_4\phi_3\left[\begin{matrix} q^{-n}, & d, & e, & aq/bc \\ & aq/b, & aq/c, & deq^{-n}/a \end{matrix}; q, q\right]. \end{aligned}$$

Thus we get

(2.20)

$$\xi_{k,r} = \left(a_2^{-1}b_3^{-1}c^{-1}q^{-r}\right)_k\left(a_2^{-1}b_2^{-1}b_3^{-1}q^{-N-r}\right)_k \cdot \frac{\left(cb_2^{-1}q^{1-N}\right)_x\left(a_1^{-1}b_1^{-1}q^{-x}\right)_x}{\left(a_1^{-1}b_2^{-1}cq^{-N}\right)_x\left(b_1^{-1}q^{1-x}\right)_x}$$

$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-k}, & a_1q, & q^{-x}, & b_1b_2^{-1}cq^{x-N} \\ & a_2b_3cq^{1+r-k}, & a_1b_1q, & a_2^{-1}b_2^{-1}b_3^{-1}q^{-N-r} \end{matrix}; q, q\right]$$

$$= \frac{(b_1)_k\left(a_2^{-1}b_3^{-1}c^{-1}q^{-r}\right)_k\left(a_1^{-1}a_2^{-1}b_2^{-1}b_3^{-1}q^{-N-1-r}\right)_k}{(a_1b_1q)_k}$$

$$\cdot \frac{\left(cb_2^{-1}q^{1-N}\right)_x\left(a_1^{-1}b_1^{-1}q^{-x}\right)_x}{\left(a_1^{-1}b_2^{-1}cq^{-N}\right)_x\left(b_1^{-1}q^{1-x}\right)_x}$$

$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-k}, & a_1q, & a_2b_3cq^{x+1+r-k}, & a_2b_2b_3b_1^{-1}q^{N-x+1+r-k} \\ & a_2b_3cq^{1+r-k}, & b_1^{-1}q^{1-k}, & a_1a_2b_2b_3q^{N+2+r-k} \end{matrix}; q, q\right].$$

In deriving the last line above we have made use of (1.5) once again. We now substitute (2.20) in (2.17), simplify the terms by means of (2.5) and (2.6) and replace the summation variable r by $r-k$. This leads to

$$L(x) = B(x)\sum_{r=0}^{n}\frac{(q^{-n})_r\left(q^{n+1}a_1a_2b_2b_3\right)_r}{(a_2b_3q)_r(b_2b_3)_r}q^r$$

$$\cdot\sum_{k=0}^{r}\sum_{l=0}^{r-k}\begin{bmatrix} r \\ k\end{bmatrix}\frac{(b_1)_{r-k}(b_2)_k(b_3)_{r-k}}{(q)_l(a_1b_1q)_{r-k}} \tag{2.21}$$

$$\cdot\frac{(a_2q)_k(q^{k-r})_l(a_1q)_l\left(a_2b_3cq^{1+x}\right)_{k+l}\left(a_2b_2b_3b_1^{-1}q^{N-x+1}\right)_{k+l}}{\left(a_2b_2b_3b_1^{-1}q\right)_k(a_2b_3cq)_{k+l}\left(b_1^{-1}q^{1-r+k}\right)_l\left(a_1a_2b_2b_3q^{N+2}\right)_{k+l}}b_3^kq^l,$$

where

$$B(x) = \frac{(q)_\infty^2}{(a_1q)_\infty(b_1)_\infty}\frac{(a_1b_1q)_x\left(cb_2^{-1}q^{1-N}\right)_x}{(q)_x\left(a_1^{-1}b_2^{-1}cq^{-N}\right)_x}(a_1q)^{-x}. \tag{2.22}$$

By changing the order of summation in (2.21) one can easily see that

$$B^{-1}(x)L(x) = \sum_{s=0}^{n}\frac{\left(a_2b_3cq^{1+x}\right)_s\left(a_2b_2b_3b_1^{-1}q^{N-x+1}\right)_s}{(a_2b_3cq)_s\left(a_1a_2b_2b_3q^{N+2}\right)_s}U_{n,s} \tag{2.23}$$

where

$$U_{n,s} = \sum_{r=s}^{n}\frac{(q^{-n})_r\left(q^{n+1}a_1a_2b_2b_3\right)_rq^{2r}}{(q)_r(a_2b_3q)_r(b_2b_3)_r}$$

$$\cdot\sum_{k=0}^{r}\begin{bmatrix} r \\ k\end{bmatrix}\frac{(b_1)_{r-k}(b_3)_{r-k}}{(a_1b_1q)_{r-k}}\frac{(b_2)_k(a_2q)_k(q^{k-r})_{s-k}(a_1q)_{s-k}}{(q)_{s-k}\left(b_1^{-1}q^{1-r+k}\right)_{s-k}\left(a_2b_2b_3b_1^{-1}q\right)_k}b_3^kq^{s-k}. \tag{2.24}$$

Simplifying the terms in (2.24) by means of (2.5) and (2.6) and changing the summation variables, we get

$$U_{n,s}=\frac{(q^{-n})_s(q^{n+1}a_1a_2b_2b_3)_s(b_2)_s(a_2q)_s}{(q)_s(a_2b_3q)_s(b_2b_3)_s(a_2b_2b_3b_1^{-1}q)_s}(b_3q)^s$$

(2.25) $$\cdot\sum_{k=0}^{n-s}\frac{(q^{s-n})_k(q^{s+n+1}a_1a_2b_2b_3)_k}{(q)_k(b_2b_3q^2)_k}\frac{(b_1)_k(b_3)_k}{(a_1b_1q)_k(a_2b_3q^{s+1})_k}q^k$$

$$\cdot{}_4\phi_3\left[\begin{matrix}q^{-s}, & a_1q, & b_3q^k, & b_1a_2^{-1}b_2^{-1}b_3^{-1}q^{-s} \\ & a_1b_1q^{k+1}, & b_2^{-1}q^{1-s}, & a_2^{-1}q^{-s}\end{matrix};q,q\right].$$

Once again the terminating ${}_4\phi_3$ series on the right turns out to be balanced. Applying (1.5) we now have

$${}_4\phi_3\left[\begin{matrix}q^{-s}, & b_3q^k, & a_1q, & b_1a_2^{-1}b_2^{-1}b_3^{-1}q^{-s} \\ & a_1b_1q^{k+1}, & b_2^{-1}q^{1-s}, & a_2^{-1}q^{-s}\end{matrix};q,q\right]$$

$$=\frac{(b_2b_3q^k)_s(a_2b_3q^{k+1})_s}{(b_2^{-1}q^{1-s})_s(a_2^{-1}q^{-s})_s}(a_2b_2b_3q^{k+s})^{-s}$$

$$\cdot{}_4\phi_3\left[\begin{matrix}q^{-s}, & q^{s+k+1}a_1a_2b_2b_3, & b_1q^k, & b_3q^k \\ & a_1b_1q^{k+1}, & b_2b_3q^k, & a_2b_3q^{k+1}\end{matrix};q,q\right]$$

(2.26) $$=\frac{(b_2b_3q^k)_s(a_2b_3q^{k+1})_s}{(b_2)_s(a_2q)_s}(b_3q^k)^{-s}$$

$$\cdot\sum_{l=0}^{s}\frac{(q^{-s})_l(q^{s+k+1}a_1a_2b_2b_3)_l(b_1q^k)_l(b_3q^k)_l}{(q)_l(a_1b_1q^{k+1})_l(b_2b_3q^k)_l(a_2b_3q^{k+1})_l}q^l,$$

by virtue of (1.11). Substituting (2.26) in (2.25) and simplifying, we obtain

$$U_{n,s}=\frac{(q^{-n})_s(q^{n+1}a_1a_2b_2b_3)_sq^s}{(q)_s(a_2b_2b_3b_1^{-1}q)_s}$$

$$\cdot\sum_{k=0}^{n-s}\sum_{l=0}^{s}\frac{(q^{s-n})_k(q^{s+n+1}a_1a_2b_2b_3)_k}{(q)_k(q)_l(q^{s+1}a_1a_2b_2b_3)_k}$$

$$\cdot\frac{(q^{-s})_l(b_1)_{k+l}(b_3)_{k+l}(q^{s+1}a_1a_2b_2b_3)_{k+l}}{(a_1b_1q)_{k+l}(a_2b_3q)_{k+l}(b_2b_3)_{k+l}}\cdot q^{k+l-ks}$$

(2.27) $$=\frac{(q^{-n})_s(q^{n+1}a_1a_2b_2b_3)_sq^s}{(q)_s(a_2b_2b_3b_1^{-1}q)_s}$$

$$\cdot\sum_{m=0}^{n}\frac{(q^{s+1}a_1a_2b_2b_3)_m(b_1)_m(b_3)_m}{(a_1b_1q)_m(a_2b_3q)_m(b_2b_3)_m}q^m$$

$$\cdot\sum_{k}\frac{(q^{-s})_{m-k}(q^{s+n+1}a_1a_2b_2b_3)_k(q^{s-n})_k}{(q)_k(q)_{m-k}(q^{s+1}a_1a_2b_2b_3)_k}q^{-ks}.$$

However, by (2.5) and (2.3),

$$\tag{2.28}\begin{aligned}\sum_k &\frac{(q^{-s})_{m-k}(q^{s+n+1}a_1a_2b_2b_3)_k(q^{s-n})_k}{(q)_k(q)_{m-k}(q^{s+1}a_1a_2b_2b_3)_k}q^{-ks}\\
&=\frac{(q^{-s})}{(q)_m}{}_3\phi_2\left[\begin{matrix}q^{-m}, & q^{s-n}, & q^{s+n+1}a_1a_2b_2b_3\\ & q^{s+1-m}, & q^{s+1}a_1a_2b_2b_3\end{matrix};q,q\right]\\
&=\frac{(q^{-s})_m}{(q)_m}\cdot\frac{(q^{-n})_m(q^{n+1}a_1a_2b_2b_3)_m}{(q^{s+1}a_1a_2b_2b_3)_m(q^{-s})_m}=\frac{(q^{-n})_m(q^{n+1}a_1a_2b_2b_3)_m}{(q)_m(q^{s+1}a_1a_2b_2b_3)_m}.\end{aligned}$$

Thus, (2.27) and (2.28) give

$$\tag{2.29}\begin{aligned}U_{n,s}&=\frac{(q^{-n})_s(q^{n+1}a_1a_2b_2b_3)_sq^s}{(q)_s(a_2b_2b_3b_1^{-1}q)_s}\\
&\cdot{}_4\phi_3\left[\begin{matrix}q^{-n}, & q^{n+1}a_1a_2b_2b_3, & b_1, & b_3\\ & b_2b_3, & a_1b_1q, & a_2b_3q\end{matrix};q,q\right].\end{aligned}$$

But the ${}_4\phi_3$ on the right is balanced and so it transforms to

$$\tag{2.30}\begin{aligned}&{}_4\phi_3\left[\begin{matrix}q^{-n}, & q^{n+1}a_1a_2b_2b_3, & b_1, & b_3\\ & b_2b_3, & a_1b_1q, & a_2b_3q\end{matrix};q,q\right]\\
&=\frac{(a_1b_2q)_n(a_2b_2b_3b_1^{-1}q)_n}{(a_1b_1q)_n(a_2b_3q)_n}\left(\frac{b_1}{b_2}\right)^n\\
&\cdot{}_4\phi_3\left[\begin{matrix}q^{-n}, & q^{n+1}a_1a_2b_2b_3, & b_2, & b_2b_3b_1^{-1}\\ & a_1b_2q, & b_2b_3, & a_2b_2b_3b_1^{-1}q\end{matrix};q,q\right].\end{aligned}$$

Thus we have

(2.31)

$$\begin{aligned}B^{-1}(x)L(x)&=\frac{(a_1b_2q)_n(a_2b_2b_3b_1^{-1}q)_n}{(a_1b_1q)_n(a_2b_3q)_n}\left(\frac{b_1}{b_2}\right)^n\\
&\cdot\lambda_n\,{}_4\phi_3\left[\begin{matrix}q^{-n}, & q^{n+1}a_1a_2b_2b_3, & a_2b_3cq^{1+x}, & a_2b_2b_3b_1^{-1}q^{N-x+1}\\ & a_2b_3cq, & a_1a_2b_2b_3q^{N+2}, & a_2b_2b_3b_1^{-1}q\end{matrix};q,q\right]\\
&=\frac{(q^{-N})_n(a_1b_2q)_n}{(a_2b_3q)_n(a_1a_2b_2b_3q^{N+2})_n}(a_2b_3q^{N+1})^n\lambda_nW_n^{(1)}(x;q).\end{aligned}$$

Note that λ_n, as defined in (1.19), is the same as the ${}_4\phi_3$ in (2.30), and $W_n^{(1)}(x;q)$ as defined in (1.16) is obtained from the ${}_4\phi_3$ of the first line in (2.31) by using the transformation (2.5) yet another time.

If we now follow the sequence of operations performed on $W_n^{(2)}(y;q)$ we find that

$$
\begin{aligned}
&[B(x)A(x)]^{-1}\sum_{i=0}^{x}\frac{\left(cb_2^{-1}q^{-N}\right)_i\left(q\sqrt{cb_2^{-1}q^{-N}}\right)_i\left(-q\sqrt{cb_2^{-1}q^{-N}}\right)_i\left(b_1b_2^{-1}cq^{x-N}\right)_i}{\left(\sqrt{cb_2^{-1}q^{-N}}\right)_i\left(-\sqrt{cb_2^{-1}q^{-N}}\right)_i\left(cb_2^{-1}q^{1+x-N}\right)_i\left(ca_1^{-1}b_2^{-1}q^{-N}\right)_i}\\
&\qquad\cdot\frac{\left(q^{i+1}\right)_\infty\left(q^{x-i+1}\right)_\infty}{\left(a_1q^{i+1}\right)_\infty\left(b_1q^{x-i}\right)_\infty}(qa_1)^{-i}\\
&\qquad\cdot\sum_{j=x}^{N}\frac{1-cb_3q^{2j-N}}{1-cb_3q^{-N}}\frac{\left(a_2b_3cq\right)_j\left(b_3cq^{x-N}\right)_j}{\left(b_3cq\right)_j\left(b_1b_2^{-1}cq^{1+x-N}\right)_j}\\
&\qquad\cdot\frac{\left(q^{j-x+1}\right)_\infty\left(q^{N-j+1}\right)_\infty\left(b_1b_2^{-1}b_3^{-1}\right)^j}{\left(b_1^{-1}b_2b_3q^{j-x}\right)_\infty\left(a_2q^{N-j+1}\right)_\infty}\\
&\qquad\cdot C^{-1}(i,j)\sum_{y=i}^{j}b_2^{-y}\frac{1-cq^{2y-N}}{1-cq^{2i-N}}\frac{\left(cq^{i-N}\right)_y\left(b_3cq^{j-N}\right)_y}{\left(cq^{1+j-N}\right)_y\left(b_2^{-1}cq^{1+i-N}\right)_y}\\
&\qquad\cdot\frac{\left(q^{y-i+1}\right)_\infty\left(q^{j-y+1}\right)_\infty}{\left(b_2q^{y-i}\right)_\infty\left(b_3q^{j-y}\right)_\infty}W_n^{(2)}(y;q)\\
&\qquad=\lambda_nW_n^{(1)}(x;q),
\end{aligned}
\tag{2.32}
$$

where $A(x)$, $B(x)$ and $C(i,j)$ are defined by (2.15), (2.22) and (2.9), respectively. The reproducing kernel defined in (2.32) is the same as the one given in (1.20) and (1.21) once we express it in terms of basic hypergeometric coefficients and make repeated use of identities (1.11), (2.5) and (2.6). However, it may be pointed out that the nonnegativity of the kernel under conditions (1.22) or (1.23) is somewhat more obvious in (2.32) than in (1.20).

3. Some special cases.

Case I: $c = 0$. As we have seen in §1 the q-Racah polynomials reduce to q-Hahn polynomials as c approaches 0. So by taking the limit $c \to 0$ in (1.18) we should get a connection relation for $Q_n(x;q)$. Since $\lim_{c\to 0}(ac)_m = 1$ and $\lim_{c\to 0}(ac^{-1})_m/(bc^{-1})_m = (a/b)^m$, the connection relation (1.18) becomes

$$\sum_{y=0}^{N} L_N(x,y)Q_n^{(2)}(y;q) = \lambda_n Q_n^{(1)}(x;q) \tag{3.1}$$

where λ_n is the same as in (1.19) and the q-Hahn polynomials are given by

$$Q_n^{(1)}(x;q) = {}_3\phi_2\left[\begin{matrix} q^{-n}, & q^{n+1}a_1a_2b_2b_3, & q^{-x} \\ & a_1b_1q, & q^{-N} \end{matrix};q,q\right], \tag{3.2}$$

$$Q_n^{(2)}(y;q) = {}_3\phi_2\left[\begin{matrix} q^{-n}, & q^{n+1}a_1a_2b_2b_3, & q^{-y} \\ & a_1b_2q, & q^{-N} \end{matrix};q,q\right], \tag{3.3}$$

while the kernel $L_N(x, y)$ is defined by

$$
\begin{aligned}
L_N(x, y) = B_N(x, y) \sum_{i=0}^{x\wedge y} \frac{(a_1q)_i(b_2^{-1}b_3^{-1}q^{1-N})_i(q^{-x})_i(q^{-y})_i}{(q)_i(b_1^{-1}q^{1-x})_i(q^{-N})_i(b_2^{-1}q^{1-y})_i}\left(\frac{b_3}{a_1b_1}\right)^i \\
\cdot {}_4\phi_3\left[\begin{matrix} a_2q, & b_2^{-1}b_3^{-1}q^{1+i-N}, & q^{x-N}, & q^{y-N} \\ & b_1b_2^{-1}b_3^{-1}q^{1+x-N}, & b_3^{-1}q^{1+y-N}, & q^{i-N} \end{matrix}; q, q\right],
\end{aligned} \tag{3.4}
$$

with

$$
B_N(x, y) = \frac{(q)_N}{(b_2b_3)_N}\frac{(b_2b_3b_1^{-1})_{N-x}(b_1)_x(b_2)_y(b_3)_{N-y}(a_1q)^x}{(a_2b_2b_3b_1^{-1}q)_{N-x}(a_1b_1q)_x(q)_y(q)_{N-y}}b_2^{N-y}. \tag{3.5}
$$

We would like to point out that this special case represents a q-extension of the results in [**8**].

Case II: $c = 0$, $a_2q = 1$. Setting $a_2q = 1$ in (1.19) we find that the eigenvalue reduces to a balanced ${}_3\phi_2$. So, using the summation formula (2.3) we get

$$
\begin{aligned}
\lambda_n &= {}_3\phi_2\left[\begin{matrix} q^{-n}, & q^na_1a_2b_2b_3, & b_2 \\ & a_1b_2q, & b_2b_3 \end{matrix}; q, q\right] \\
&= \frac{(a_1q)_n(b_3^{-1}q^{1-n})_n}{(a_1b_2q)_n(b_2^{-1}b_3^{-1}q^{1-n})_n} = \frac{(a_1q)_n(b_3)}{(a_1b_2q)_n(b_2b_3)_n}b_2^n
\end{aligned} \tag{3.6}
$$

by (1.11). Hence the connection relation reduces to

$$
\begin{aligned}
&\sum_{y=0}^{N} M_N(x, y){}_3\phi_2\left[\begin{matrix} q^{-n}, & q^na_1b_2b_3, & q^{-y} \\ & a_1b_2q, & q^{-N} \end{matrix}; q, q\right] \\
&= \frac{(a_1q)_n(b_3)_nb_2^n}{(a_1b_2q)_n(b_2b_3)_n}{}_3\phi_2\left[\begin{matrix} q^{-n}, & q^na_1b_2b_3, & q^{-x} \\ & a_1b_1q, & q^{-N} \end{matrix}; q, q\right],
\end{aligned} \tag{3.7}
$$

where

$$
\begin{aligned}
M_N(x, y) = \frac{(q)_N(b_1)_x(b_2)_y(b_3)_{N-y}}{(b_2b_3)_N(a_1b_1q)_x(q)_y(q)_{N-y}}(a_1q)^xb_2^{N-y} \\
\cdot {}_4\phi_3\left[\begin{matrix} a_1q, & b_2^{-1}b_3^{-1}q^{1-N}, & q^{-x}, & q^{-y} \\ & b_1^{-1}q^{1-x}, & q^{-N}, & b_2^{-1}q^{1-y} \end{matrix}; q, \frac{b_3}{a_1b_1}\right].
\end{aligned} \tag{3.8}
$$

Case III: $c = 0$, $a_1q = 1$. A similar calculation leads to the connection relation

$$
\begin{aligned}
&\sum_{y=0}^{N} P_N(x, y){}_3\phi_2\left[\begin{matrix} q^{-n}, & q^na_2b_2b_3, & q^{-y} \\ & b_1, & q^{-N} \end{matrix}; q, q\right] \\
&= \frac{(b_1)_n(a_2q)}{(b_2b_3)_n(a_2b_2b_3b_1^{-1}q)_n}\left(\frac{b_2b_3}{b_1}\right)^n{}_3\phi_2\left[\begin{matrix} q^{-n}, & q^na_2b_2b_3, & q^{-x} \\ & b_2, & q^{-N} \end{matrix}; q, q\right],
\end{aligned} \tag{3.9}
$$

where

$$(3.10)\quad P_N(x,y)=\frac{(q)_N(b_2b_3b_1^{-1})_{N-x}(b_2)_y(b_3)_{N-y}}{(b_2b_3)_N(a_2b_2b_3b_1^{-1}q)_{N-x}(q)_y(q)_{N-y}}b_2^{N-y}$$

$$\cdot{}_4\phi_3\left[\begin{matrix}a_2q, & b_2^{-1}b_3^{-1}q^{1-N}, & q^{x-N}, & q^{y-N}\\ & b_1b_2^{-1}b_3^{-1}q^{1+x-N}, & b_3^{-1}q^{1+y-N}, & q^{-N}\end{matrix};q,q\right].$$

This case yields an interesting kernel for the little q-Jacobi polynomials defined in (1.15). If we take the limits $x, y, N\to\infty$ so that $N-x=r$ and $N-y=s$, r, s being nonnegative integers, then (3.9) becomes

$$(3.11)\quad \sum_{s=0}^{\infty}P_\infty(r,s)\,{}_2\phi_1\left[\begin{matrix}q^{-n}, & q^na_2b_2b_3,\\ & b_2\end{matrix};q,q^{s+1}\right]$$

$$=\frac{(b_1)_n(a_2q)_n}{(b_2b_3)_n(a_2b_2b_3b_1^{-1}q)_n}\left(\frac{b_2b_3}{b_1}\right)^n{}_2\phi_1\left[\begin{matrix}q^{-n}, & q^na_2b_2b_3\\ & b_2\end{matrix};q,q^{r+1}\right]$$

with

$$(3.12)\quad P_\infty(r,s)=\frac{(b_2)_\infty}{(b_2b_3)_\infty}\frac{(b_2b_3b_1^{-1})_r(b_3)_s}{(a_2b_2b_3b_1^{-1}q)_r(q)_s}b_2^s$$

$$\cdot{}_3\phi_2\left[\begin{matrix}q^{-r}, & q^{-s}, & a_2q\\ & b_1b_2^{-1}b_3^{-1}q^{1-r}, & b_3^{-1}q^{1-s}\end{matrix};q,b_2^{-1}b_3^{-1}q^2\right].$$

In the symmetric case $b_1=b_2$, this is essentially the same kernel as found by Al-Salam and Ismail [**1**] by an entirely different method.

Case IV: $a_1q=1$, $c\neq 0$. The eigenvalue λ_n is the same as in (3.9) while the "eigenfunctions" are:

$$W_n^{(1)}(x;q)={}_4\phi_3\left[\begin{matrix}q^{-n}, & q^na_2b_2b_3, & q^{-x}, & b_1b_2^{-1}cq^{x-N}\\ & b_1, & q^{-N}, & a_2b_3cq\end{matrix};q,q\right],$$

$$W_n^{(2)}(y;q)={}_4\phi_3\left[\begin{matrix}q^{-n}, & q^na_2b_2b_3, & q^{-y}, & cq^{y-N}\\ & b_2, & q^{-N}, & a_2b_3cq\end{matrix};q,q\right].$$

The kernel $K_N(x,y)$ becomes a multiple of a single ${}_{10}\phi_9$ series:

$$K_N(x,y)=\frac{(q)_N(a_2b_3cq)_N}{(b_2b_3)_N(b_3c)_N}\cdot\frac{(b_3c)_x(b_2)_y(b_3c)_y}{(a_2b_3cq)_x(q)_y(cq)_y}$$

$$\cdot\frac{(b_2b_3b_1^{-1})_{N-x}(b_3)_{N-y}(b_2c)_{N-y}}{(a_2b_2b_3b_1^{-1}q)_{N-x}(q)_{N-y}(c^{-1})_{N-y}}\frac{1-cq^{2y-N}}{1-cq^{y-N}}$$

$$(3.13)\quad \cdot{}_{10}\phi_9\left[\begin{matrix}b_3^{-1}c^{-1}q^{-N}, & q\sqrt{b_3^{-1}c^{-1}q^{-N}}, & -q\sqrt{b_3^{-1}c^{-1}q^{-N}}, & a_2q,\\ & \sqrt{b_3^{-1}c^{-1}q^{-N}}, & -\sqrt{b_3^{-1}c^{-1}q^{-N}}, & a_2^{-1}b_3^{-1}c^{-1}q^{-N},\end{matrix}\right.$$

$$\begin{matrix}b_1^{-1}b_2c^{-1}q^{-x}, & c^{-1}q^{-y}, & b_2^{-1}b_3^{-1}q^{1-N},\\ b_1b_2^{-1}b_3^{-1}q^{1+x-N}, & b_3^{-1}q^{1+y-N}, & b_2c^{-1},\end{matrix}$$

$$\left.\begin{matrix}b_3^{-1}c^{-1}q, & q^{x-N}, & q^{y-N}\\ q^{-N}, & b_3^{-1}c^{-1}q^{1-x}, & b_3^{-1}c^{-1}q^{1-y}\end{matrix};q,\frac{b_1}{a_2b_3}\right].$$

Although the above series is very well-poised, there is no known transformation formula for it unless it is also balanced, which requires that a_2q must equal b_1/b_3. Apart from Jackson's formula for a very well-poised balanced ${}_{10}\phi_9$ [**11**, equation (3.4.2.4), p. 102] the author recently derived a q-extension of Bailey's transformation for a 2-balanced very well-poised ${}_9F_8$ which reads [**10**]:

$$
\begin{aligned}
&{}_{10}\phi_9\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, \end{matrix}\right. \\
&\qquad\qquad\left.\begin{matrix} B, & C, & D, & q^{-k} \\ aq/B, & aq/C, & aq/D, & aq^{k+1} \end{matrix}; q, q\right] \\
(3.14)\quad &= \frac{(aq)_k(aq/BC)_k(aq/BD)_k(aq/Bc)_k(aq/Bd)_k(b)_k}{(aq/c)_k(aq/d)_k(aq/B)_k(aq/C)_k(aq/D)_k(b/B)_k} B^k \\
&\cdot {}_{10}\phi_9\left[\begin{matrix} Bb^{-1}q^{-k}, & q\sqrt{Bb^{-1}q^{-k}}, & -q\sqrt{Bb^{-1}q^{-k}}, & B, & aq/bc, & aq/bd, \\ & \sqrt{Bb^{-1}q^{-k}}, & -\sqrt{Bb^{-1}q^{-k}}, & b^{-1}q^{1-k}, & a^{-1}Bcq^{-k}, & a^{-1}Bdq^{-k}, \end{matrix}\right. \\
&\qquad\qquad\left.\begin{matrix} aq/bC, & aq/bD, & a^{-1}Bq^{-k}, & q^{-k} \\ a^{-1}BCq^{-k}, & a^{-1}BDq^{-k}, & b^{-1}aq, & b^{-1}Bq \end{matrix}; q, q\right],
\end{aligned}
$$

where k is a nonnegative integer, and balanced property requires

$$
(3.15)\qquad a^3q^2 = bcd\,BCDq^{-k}.
$$

Setting $a_2q = b_1/b_3$ and choosing it as B, $b_3^{-1}c^{-1}q$ as b, then applying (3.14) in (3.13) and simplifying the coefficients by means of (1.11), (2.5) and (2.6) we then obtain

$$
\begin{aligned}
(3.16)\quad K_N(x, y) &= \frac{(q)_N}{(b_2b_3)_N}\frac{(b_3c)_x(b_2)_y(b_1c)_y}{(b_1c)_x(q)_y(cq)_y} \\
&\cdot \frac{(b_1)_{N-y}(b_2b_3b_1^{-1}c^{-1})_{N-y}(b_3^{-1}c^{-1}q)_{N-y}(b_1^{-1}c^{-1}q^{1-x})_{N-y}}{(q)_{N-y}(c^{-1})_{N-y}(b_1^{-1}c^{-1}q)_{N-y}(b_3^{-1}c^{-1}q^{1-x})_{N-y}} \\
&\cdot \frac{(b_2b_3b_1^{-1})_{y-x}}{(b_2)_{y-x}} \cdot \frac{1-cq^{2y-N}}{1-cq^{y-N}} \\
&\cdot {}_{10}\phi_9\left[\begin{matrix} b_1cq^{y-N-1}, & q\sqrt{b_1cq^{y-N-1}}, & -q\sqrt{b_1cq^{y-N-1}}, \\ & \sqrt{b_1cq^{y-N-1}}, & -\sqrt{b_1cq^{y-N-1}}, \end{matrix}\right. \\
&\qquad \begin{matrix} b_1b_3^{-1}, & q^{-x} & cq^{y-N}, \\ b_3cq^{y-N}, & b_1cq^{x+y-N}, & b_1, \end{matrix} \\
&\qquad\left.\begin{matrix} b_2b_3q^{-1}, & b_1b_2^{-1}cq^{x-N}, & b_1cq^y, & q^{y-N} \\ b_1b_2^{-1}b_3^{-1}cq^{y-N+1}, & b_2q^{y-x}, & q^{-N}, & b_1c \end{matrix}; q, q\right],
\end{aligned}
$$

where we have assumed, for the sake of definiteness that $0 \leqslant x \leqslant y \leqslant N$. A similar expression can be derived when $x \geqslant y$.

An interesting situation arises if we set $b_1 b_3^{-1} = q^{-m}$ where m is a nonnegative integer, $0 \leq m \leq N$. Then

$$(3.17)\qquad \begin{aligned} W_n^{(1)}(x;q) &= {}_4\Phi_3\left[\begin{matrix} q^{-n}, & q^{n-1}b_1b_2, & q^{-x}, & b_1b_2^{-1}cq^{x-N} \\ & b_1, & q^{-N}, & b_1c \end{matrix}; q, q\right], \\ W_n^{(2)}(y;q) &= {}_4\Phi_3\left[\begin{matrix} q^{-n}, & q^{n-1}b_1b_2, & q^{-y}, & cq^{y-N} \\ & b_2, & q^{-N}, & b_1c \end{matrix}; q, q\right], \end{aligned}$$

$$(3.18)\qquad \lambda_n = \frac{(q^{-m})_n (b_1)_n}{(b_2)_n (b_1b_2q^m)_n} (b_2q^m)^n,$$

and

$$\begin{aligned} K_N(x,y) &= \frac{(q)_N}{(b_1b_2q^m)_N} \frac{(b_1cq^m)_x (b_2)_y (b_1c)_y}{(b_1c)_x (q)_y (cq)_y} \\ &\cdot \frac{(b_1)_{N-y}(b_2c^{-1}q^m)_{N-y}(b_1^{-1}c^{-1}q^{1-m})_{N-y}(b_1^{-1}c^{-1}q^{1-x})_{N-y}}{(q)_{N-y}(c^{-1})_{N-y}(b_1^{-1}c^{-1}q)_{N-y}(b_1^{-1}c^{-1}q^{1-m-x})_{N-y}} \\ (3.19)\qquad &\cdot \frac{(b_2q^m)_{y-x}}{(b_2)_{y-x}} \frac{1-cq^{2y-N}}{1-cq^{y-N}} \\ &\cdot {}_{10}\Phi_9\left[\begin{matrix} b_1cq^{y-N-1}, & q\sqrt{b_1cq^{y-N-1}}, & -q\sqrt{b_1cq^{y-N-1}}, & q^{-m}, \\ & \sqrt{b_1cq^{y-N-1}}, & -\sqrt{b_1cq^{y-N-1}}, & b_1cq^{m+y-N}, \end{matrix}\right. \\ &\qquad \left.\begin{matrix} b_1b_2q^{m-1}, & q^{-x}, & cq^{y-N}, & b_1b_2^{-1}cq^{x-N}, & b_1cq^y, & q^{y-N} \\ b_2^{-1}cq^{y-N-m+1}, & b_1cq^{x+y-N}, & b_1, & b_2q^{y-x}, & q^{-N}, & b_1c \end{matrix}; q, q\right]. \end{aligned}$$

Case V: $a_2q = 1$, $c \neq 0$. Here the eigenvalue λ_n is the same as in (3.6) and the eigenfunctions are

$$\begin{aligned} W_n^{(1)}(x;q) &= {}_4\Phi_3\left[\begin{matrix} q^{-n}, & q^n a_1b_2b_3, & q^{-x}, & b_1b_2^{-1}cq^{x-N} \\ & a_1b_1q, & q^{-N}, & b_3c \end{matrix}; q, q\right], \\ W_n^{(2)}(y,q) &= {}_4\Phi_3\left[\begin{matrix} q^{-n}, & q^n a_1b_2b_3, & q^{-y}, & cq^{y-N} \\ & a_1b_2q, & q^{-N}, & b_3c \end{matrix}; q, q\right]. \end{aligned}$$

Also

$$\begin{aligned} K_N(x,y) &= \frac{(q)_N(a_1b_2c^{-1}q)_N}{(b_2b_3)_N(b_2c^{-1})_N} \cdot \frac{(b_1)_x(b_2)_y(b_3c)_y}{(a_1b_1q)_x(q)_y(cq)_y} \\ &\cdot \frac{(b_2c^{-1})_{N-x}(b_3)_{N-y}(b_2c^{-1})_{N-y}}{(a_1b_2c^{-1}q)_{N-x}(q)_{N-y}(c^{-1})_{N-y}} \cdot \frac{1-cq^{2y-N}}{1-cq^{y-N}} \\ (3.20)\qquad &\cdot {}_{10}\Phi_9\left[\begin{matrix} cb_2^{-1}q^{-N}, & q\sqrt{cb_2^{-1}q^{-N}}, & -q\sqrt{cb_2^{-1}q^{-N}}, & a_1q, \\ & \sqrt{cb_2^{-1}q^{-N}}, & -\sqrt{cb_2^{-1}q^{-N}}, & ca_1^{-1}b_2^{-1}q^{-N}, \end{matrix}\right. \\ &\qquad \begin{matrix} b_1b_2^{-1}cq^{x-N}, & b_2^{-1}b_3^{-1}q^{1-N}, & cb_2^{-1}q, & cq^{y-N}, \\ b_1^{-1}q^{1-x}, & b_3c, & q^{-N}, & b_2^{-1}q^{1-y}, \end{matrix} \\ &\qquad\qquad \left.\begin{matrix} q^{-x}, & q^{-y} \\ cb_2^{-1}q^{1+x-N}, & cb_2^{-1}q^{1+y-N} \end{matrix}; q; \frac{b_3}{a_1b_1}\right]. \end{aligned}$$

The ${}_{10}\phi_9$ series is balanced if $a_1q = b_3/b_1$. As in the previous case it can be transformed by (3.14) if this condition is satisfied. If, further, $b_3/b_1 = q^{-m}$, m a nonnegative integer, then the kernel $K_N(x, y)$ above assumes a form similar to that in (3.19).

4. Bilinear formulas. Let us now multiply equation (1.18) by

$$\frac{(a_1a_2b_2b_3q)_n\left(q\sqrt{a_1a_2b_2b_3q}\right)_n\left(-q\sqrt{a_1a_2b_2b_3q}\right)_n(a_1b_2q)_n(a_2b_3cq)_n(q^{-N})_n}{(q)_n\left(\sqrt{a_1a_2b_2b_3q}\right)_n\left(-\sqrt{a_1a_2b_2b_3q}\right)_n(a_2b_3q)_n\left(a_1b_2c^{-1}q\right)_n\left(a_1a_2b_2b_3q^{N+2}\right)_n}$$
$$\cdot\left(c^{-1}q^N\right)^nW_n^{(2)}(z;q)$$

and sum over n from 0 to N. Then, by the dual orthogonality relation (1.12) we obtain, after some simplification and replacing z by y, the following bilinear formula:

$$\begin{aligned}
&\frac{(a_2b_3cq)_N(a_1a_2b_2b_3q^2)_N\left(a_1b_2c^{-1}q\right)_N}{(b_3c)_N(b_2b_3)_N\left(b_2c^{-1}\right)_N}\\
&\cdot\frac{(b_1)_x(b_3c)_x\left(b_2c^{-1}\right)_{N-x}\left(b_2b_3b_1^{-1}\right)_{N-x}}{(a_1b_1q)_x(a_2b_3cq)_x\left(a_1b_2c^{-1}q\right)_{N-x}\left(a_2b_2b_3b_1^{-1}q\right)_{N-x}}\\
&\cdot\frac{(b_2)_y(b_3c)_y\left(b_2c^{-1}\right)_{N-y}(b_3)_{N-y}}{(a_1b_2q)_y(a_2b_3cq)_y\left(a_1b_2c^{-1}q\right)_{N-y}(a_2b_3q)_{N-y}}\\
&\cdot\sum_{i=0}^{x\wedge y}\frac{\left(cb_2^{-1}q^{-N}\right)_i\left(q\sqrt{cb_2^{-1}q^{-N}}\right)_i\left(-q\sqrt{cb_2^{-1}q^{-N}}\right)_i(a_1q)_i}{(q)_i\left(\sqrt{cb_2^{-1}q^{-N}}\right)_i\left(-\sqrt{cb_2^{-1}q^{-N}}\right)_i\left(ca_1^{-1}b_2^{-1}q^{-N}\right)_i}\\
&\cdot\frac{\left(b_1b_2^{-1}cq^{x-N}\right)_i\left(b_2^{-1}b_3^{-1}q^{1-N}\right)_i\left(cb_2^{-1}q\right)_i}{\left(b_1^{-1}q^{1-x}\right)_i(b_3c)_i(q^{-N})_i\left(b_2^{-1}q^{1-y}\right)_i}\\
&\cdot\frac{\left(cq^{y-N}\right)_i(q^{-x})_i(q^{-y})_i}{\left(cb_2^{-1}q^{1+x-N}\right)_i\left(cb_2^{-1}q^{1+y-N}\right)_i}\left(\frac{b_3}{a_1b_1}\right)^i\\
&\cdot{}_{10}\phi_9\left[\begin{matrix}b_3^{-1}c^{-1}q^{-N}, & q\sqrt{b_3^{-1}c^{-1}q^{-N}}, & -q\sqrt{b_3^{-1}c^{-1}q^{-N}}, & a_2q,\\ & \sqrt{b_3^{-1}c^{-1}q^{-N}}, & -\sqrt{b_3^{-1}c^{-1}q^{-N}}, & a_2^{-1}b_3^{-1}c^{-1}q^{-N},\end{matrix}\right.\\
&\qquad\begin{matrix}b_1^{-1}b_2c^{-1}q^{-x}, & c^{-1}q^{-y}, & b_2^{-1}b_3^{-1}q^{1+i-N},\\ b_1b_2^{-1}b_3^{-1}q^{1+x-N}, & b_3^{-1}q^{1+y-N}, & b_2c^{-1}q^{-i},\end{matrix}\\
&\qquad\left.\begin{matrix}b_3^{-1}c^{-1}q^{1-i} & q^{x-N} & q^{y-N}\\ q^{i-N} & b_3^{-1}c^{-1}q^{1-x}, & b_3^{-1}c^{-1}q^{1-y}\end{matrix};q,\frac{b_1}{a_2b_3}\right]\\
&=\sum_{n=0}^{N}\frac{(a_1a_2b_2b_3q)_n\left(q\sqrt{a_1a_2b_2b_3q}\right)_n\left(-q\sqrt{a_1a_2b_2b_3q}\right)_n}{(q)_n\left(\sqrt{a_1a_2b_2b_3q}\right)_n\left(-\sqrt{a_1a_2b_2b_3q}\right)_n(a_2b_3q)_n}\\
&\cdot\frac{(a_1b_2q)_n(a_2b_3cq)_n(q^{-N})_n}{\left(a_1b_2c^{-1}q\right)_n\left(a_1a_2b_2b_3q^{N+2}\right)_n}\cdot\left(c^{-1}q^N\right)^n\lambda_nW_n^{(1)}(x;q)W_n^{(2)}(y;q).
\end{aligned}\tag{4.1}$$

A number of identities can be deduced from this, but we shall be interested mainly in those that correspond to the special cases discussed in the previous section.

First of all, in the limit $c \to 0$, we get the following bilinear formula for the q-Hahn polynomials:

$$
\begin{aligned}
&\frac{(a_1a_2b_2b_3q^2)_N}{(b_2b_3)_N}\,\frac{(b_1)_x(b_2b_3b_1^{-1})_{N-x}(b_2)_y(b_3)_{N-y}}{(a_1b_1q)_x(a_2b_2b_3b_1^{-1}q)_{N-x}(a_1b_2q)_y(a_2b_3q)_{N-y}}(a_1q)^{x+y-N}\\
&\qquad\cdot\sum_{i=0}^{x\wedge y}\frac{(a_1q)_i(b_2^{-1}b_3^{-1}q^{1-N})_i(q^{-x})_i(q^{-y})_i}{(q)_i(b_1^{-1}q^{1-x})_i(q^{-N})_i(b_2^{-1}q^{1-y})_i}\left(\frac{b_3}{a_1b_1}\right)^i\\
&\qquad\cdot{}_4\phi_3\left[\begin{matrix}a_2q, & b_2^{-1}b_3^{-1}q^{1+i-N}, & q^{x-N}, & q^{y-N}\\ & b_1b_2^{-1}b_3^{-1}q^{1+x-N}, & b_3^{-1}q^{1+y-N}, & q^{i-N}\end{matrix};q,q\right]\\
&=\sum_{n=0}^{N}\frac{(a_1a_2b_2b_3q)_n\left(q\sqrt{a_1a_2b_2b_3q}\right)_n\left(-\sqrt{a_1a_2b_2b_3q}\right)_n(a_1b_2q)_n(q^{-N})_n}{(q)_n\left(\sqrt{a_1a_2b_2b_3q}\right)_n\left(-\sqrt{a_1a_2b_2b_3q}\right)_n(a_2b_3q)_n(a_1a_2b_2b_3q^{N+2})_n}\left(-\frac{q^N}{a_1b_2}\right)^n\\
&\qquad\cdot q^{-n(n+1)/2}\cdot\lambda_n\,{}_3\phi_2\left[\begin{matrix}q^{-n}, & q^{n+1}a_1a_2b_2b_3, & q^{-x}\\ & a_1b_1q, & q^{-N}\end{matrix};q,q\right]\\
&\qquad\cdot{}_3\phi_2\left[\begin{matrix}q^{-n}, & q^{n+1}a_1a_2b_2b_3, & q^{-y}\\ & a_1b_2q, & q^{-N}\end{matrix};q,q\right].
\end{aligned}\tag{4.2}
$$

This may be viewed as a q-extension of the bilinear formula that we found in **[8]** for Hahn polynomials. Setting $a_1q = 1$ in (4.2) we get a much simpler formula:

$$
\begin{aligned}
&\frac{(a_2b_2b_3q)_N}{(b_2b_3)_N}\,\frac{(b_2b_3b_1^{-1})_{N-x}(b_3)_{N-y}}{(a_2b_2b_3b_1^{-1}q)_{N-x}(a_2b_3q)_{N-y}}\\
&\qquad\cdot{}_4\phi_3\left[\begin{matrix}a_2q, & b_2^{-1}b_3^{-1}q^{1-N}, & q^{x-N}, & q^{y-N}\\ & b_1b_2^{-1}b_3^{-1}q^{1+x-N}, & b_3^{-1}q^{1+y-N}, & q^{-N}\end{matrix};q,q\right]\\
&=\sum_{n=0}^{N}\frac{(a_2b_2b_3)_n\left(q\sqrt{a_2b_2b_3}\right)_n\left(-q\sqrt{a_2b_2b_3}\right)_n(b_1)_n(b_2)_n(a_2q)_n(q^{-N})_n}{(q)_n\left(\sqrt{a_2b_2b_3}\right)_n\left(-\sqrt{a_2b_2b_3}\right)_n(a_2b_2b_3b_1^{-1}q)_n(a_2b_3q)_n(b_2b_3)_n(a_2b_2b_3q^{N+2})_n}\\
&\qquad\cdot\left(-b_1^{-1}b_3q^{1+N}\right)^n\cdot q^{-n(n+1)/2}\,{}_3\phi_2\left[\begin{matrix}q^{-n}, & q^na_2b_2b_3, & q^{-x}\\ & b_1, & q^{-N}\end{matrix};q,q\right]\\
&\qquad\cdot{}_3\phi_2\left[\begin{matrix}q^{-n}, & q^na_2b_2b_3, & q^{-y}\\ & b_2, & q^{-N}\end{matrix};q,q\right].
\end{aligned}\tag{4.3}
$$

We may now take the Jacobi limit: $x, y, N \to \infty$ with $N - x = r$, $N - y = s$, $r, s = 0, 1, 2, \ldots$. Then (4.3) reduces to

$$
\begin{aligned}
&\frac{(a_2b_2b_3q)_\infty}{(b_2b_3)_\infty}\,\frac{(b_2b_3b_1^{-1})_r(b_3)_s}{(a_2b_2b_3b_1^{-1}q)_r(a_2b_3q)_s}\\
&\qquad\cdot{}_3\phi_2\left[\begin{matrix}a_2q, & q^{-r}, & q^{-s}\\ & b_1b_2^{-1}b_3^{-1}q^{1-r}, & b_3^{-1}q^{1-s}\end{matrix};q,b_2^{-1}b_3^{-1}q^2\right]\\
&=\sum_{n=0}^{\infty}\frac{(a_2b_2b_3)_n\left(q\sqrt{a_2b_2b_3}\right)_n\left(-q\sqrt{a_2b_2b_3}\right)_n(b_1)_n(b_2)_n(a_2q)_n}{(q)_n\left(\sqrt{a_2b_2b_3}\right)_n\left(-\sqrt{a_2b_2b_3}\right)_n(a_2b_2b_3b_1^{-1}q)_n(a_2b_3q)_n(b_2b_3)_n}\left(b_1^{-1}b_3\right)^n\\
&\qquad\cdot{}_2\phi_1\left[\begin{matrix}q^{-n}, & q^na_2b_2b_3\\ & b_1\end{matrix};q,q^{r+1}\right]{}_2\phi_1\left[\begin{matrix}q^{-n}, & q^na_2b_2b_3\\ & b_2\end{matrix};q,q^{s+1}\right].
\end{aligned}\tag{4.4}
$$

When $b_1 = b_2$, equation (4.4) leads to Al-Salam and Ismail's bilinear formula [**1**, equation (3.11)]. The validity of (4.4) cannot, of course, be taken for granted since we have an infinite series on the r.h.s. However, for $0 < b_1, b_2, b_3, b_2b_3/b_1 < 1$ and $0 < a_2q < 1$ one can prove in much the same way as done in [**1**] that the kernel on the l.h.s. is square-integrable, which ensures the validity of (4.4).

Let us now discuss the special cases of (4.1) when $c \neq 0$ and either $a_1q = 1$ or $a_2q = 1$. To fix ideas let us take $a_1q = 1$, since the case $a_2q = 1$ will give essentially the same identities. Setting $a_1q = 1$ in (4.1) we obtain

$$\frac{(a_2b_2b_3q)_N(a_2b_3cq)_N}{(b_2b_3)_N(b_3c)_N}\frac{(b_3c)_x(b_2b_3b_1^{-1})_{N-x}(b_3c)_y(b_3)_{N-y}}{(a_2b_3cq)_x(a_2b_2b_3b_1^{-1}q)_{N-x}(a_2b_3cq)_y(a_2b_3q)_{N-y}}$$
$$\cdot {}_{10}\Phi_9\left[\begin{matrix} b_3^{-1}c^{-1}q^{-N}, & q\sqrt{b_3^{-1}c^{-1}q^{-N}}, & -q\sqrt{b_3^{-1}c^{-1}q^{-N}}, & a_2q, & b_1^{-1}b_2c^{-1}q^{-x}, \\ & \sqrt{b_3^{-1}c^{-1}q^{-N}}, & -\sqrt{b_3^{-1}c^{-1}q^{-N}}, & a_2^{-1}b_3^{-1}c^{-1}q^{-N}, & b_1b_2^{-1}b_3^{-1}q^{1+x-N}, \end{matrix}\right.$$
$$\left.\begin{matrix} c^{-1}q^{-y}, & b_2^{-1}b_3^{-1}q^{1-N}, & b_3^{-1}c^{-1}q, & q^{x-N}, & q^{y-N} \\ b_3^{-1}q^{1+y-N}, & b_2c^{-1}, & q^{-N}, & b_3^{-1}c^{-1}q^{1-x}, & b_3^{-1}c^{-1}q^{1-y} \end{matrix}; q, \frac{b_1}{a_2b_3}\right] \tag{4.5}$$
$$= \sum_{n=0}^{N} \frac{(a_2b_2b_3)_n\left(q\sqrt{a_2b_2b_3}\right)_n\left(-q\sqrt{a_2b_2b_3}\right)_n(b_1)_n(b_2)_n(a_2q)_n}{(q)_n\left(\sqrt{a_2b_2b_3}\right)_n\left(-\sqrt{a_2b_2b_3}\right)_n(a_2b_2b_3b_1^{-1}q)_n(a_2b_3q)_n(b_2b_3)_n}$$
$$\cdot\frac{(a_2b_3cq)_n(q^{-N})_n}{(b_2c^{-1})_n(a_2b_2b_3q^{N+1})_n}\left(\frac{b_2b_3q^N}{b_1c}\right)^n W_n^{(1)}(x;q)W_n^{(2)}(y;q),$$

assuming that we have set $a_1q = 1$ in the expressions for $W_n^{(1)}(x; q)$ and $W_n^{(2)}(y; q)$.

This leads to a particularly interesting bilinear formula if we specialize the parameters a_2 and b_3 by setting

$$a_2q = q^{-m}, \qquad b_3 = q^m b_4, \tag{4.6}$$

where b_4 is an arbitrary parameter and m a nonnegative integer. Then, whether or not x, y, N are nonnegative integers as had been originally assumed, we get the identity

$$\frac{(b_2b_4)_m(cb_4q^x)_m(cb_4q^y)_m(b_2b_1^{-1}b_4q^{N-x})_m(b_4q^{N-y})_m}{(b_2b_4q^N)_m(cb_4q^N)_m(b_4)_m(cb_4)_m(b_2b_1^{-1}b_4)_m}$$
$$\cdot {}_{10}\Phi_9\left[\begin{matrix} b_4^{-1}c^{-1}q^{-N-m}, & q\sqrt{b_4^{-1}c^{-1}q^{-N-m}}, & -q\sqrt{b_4^{-1}c^{-1}q^{-N-m}}, & b_2^{-1}b_4^{-1}q^{1-m-N}, \\ & \sqrt{b_4^{-1}c^{-1}q^{-N-m}}, & \sqrt{b_4^{-1}c^{-1}q^{-N-m}}, & b_2c^{-1}, \end{matrix}\right.$$
$$\begin{matrix} b_4^{-1}c^{-1}q^{1-m}, & q^{-m}, & b_1^{-1}b_2c^{-1}q^{-x}, & c^{-1}q^{-y}, \\ q^{-N}, & b_4^{-1}c^{-1}q^{1-N}, & b_1b_2^{-1}b_4^{-1}q^{1+x-m-N}, & b_4^{-1}q^{1+y-N-m}, \end{matrix}$$
$$\left.\begin{matrix} q^{x-N}, & q^{y-N} \\ b_4^{-1}c^{-1}q^{1-x-m}, & b_4^{-1}c^{-1}q^{1-y-m} \end{matrix}; q, b_1b_4^{-1}q\right] \tag{4.7}$$
$$= \sum_{n=0}^{m} \frac{(b_2b_4q^{-1})_n\left(q\sqrt{b_2b_4q^{-1}}\right)_n\left(-q\sqrt{b_2b_4q^{-1}}\right)_n(b_1)_n(b_2)_n(b_4c)_n(q^{-N})_n(q^{-m})_n}{(q)_n\left(\sqrt{b_2b_4q^{-1}}\right)_n\left(\sqrt{b_2b_4q^{-1}}\right)_n(b_2b_1^{-1}b_4)_n(b_4)_n(b_2c^{-1})_n(b_2b_4q^N)_n(b_2b_4q^m)_n}$$
$$\cdot\left(\frac{b_2b_4}{cb_1}q^{N+m}\right)^n W_n^{(1)}(x;q)W_n^{(2)}(y;q).$$

By using the summation formula [**11**, p. 247]

(4.8) $$ {}_2\phi_1[a, q^{-m}; b; q, q) = (b/a)_m a^m/(b)_m, $$

we can now invert this formula if we multiply both sides by $q^m(q^{-k})_m(q^{k-1}b_2b_4)_m/(q)_m(b_2b_4)_m$ and sum over m from 0 to k. Thus we obtain

(4.9)
$$ \frac{(b_1)_k(b_2)_k(b_4c)_k(q^{-N})_k}{(b_2b_1^{-1}b_4)_k(b_4)_k(b_2c^{-1})_k(b_2b_4q^N)_k}\left(\frac{b_2b_4}{b_1c}q^N\right)^k W_k^{(1)}(x;q)W_k^{(2)}(y;q) $$
$$ = \sum_{m=0}^{k}\frac{(q^{-k})_m(q^{k-1}b_2b_4)_m(b_4cq^x)_m(b_4cq^y)_m(b_2b_1^{-1}b_4q^{N-x})_m(b_4q^{N-y})_m}{(q_m)(b_4)_m(b_2b_1^{-1}b_4)_m(b_2b_4q^N)_m(b_4cq^N)_m(b_4c)_m}q^m\cdot{}_{10}\phi_9[\;], $$

where ${}_{10}\phi_9[\;]$ is the same as the one on the l.h.s. of (4.7). This formula simplifies even further if we apply the transformation formula (1.5) on both $W_k^{(1)}(x;q)$ and $W_k^{(2)}(y;q)$:

$$ W_k^{(1)}(x;q) \equiv {}_4\phi_3\left[\begin{matrix} q^{-k}, & q^{k-1}b_2b_4, & q^{-x}, & b_1b_2^{-1}cq^{x-N} \\ & q^{-N}, & b_1, & b_4c \end{matrix}; q, q\right] $$
$$ = \frac{(b_2b_1^{-1}b_4)_k(b_2c^{-1})_k}{(b_1)_k(b_4c)_k}(b_1b_2^{-1}c)^k $$
$$ \cdot{}_4\phi_3\left[\begin{matrix} q^{-k}, & q^{k-1}b_2b_4, & q^{x-N}, & b_1^{-1}b_2c^{-1}q^{-x} \\ & b_2b_1^{-1}b_4, & q^{-N}, & b_2c^{-1} \end{matrix}; q, q\right], $$

$$ W_k^{(2)}(y;q) \equiv {}_4\phi_3\left[\begin{matrix} q^{-k}, & q^{k-1}b_2b_4, & q^{-y}, & cq^{y-N} \\ & q^{-N}, & b_2, & b_4c \end{matrix}; q, q\right] $$
$$ = \frac{(b_4)_k(b_2c^{-1})_k}{(b_2)_k(b_4c)_k}c^k\,{}_4\phi_3\left[\begin{matrix} q^{-k}, & q^{k-1}b_2b_4, & q^{y-N}, & c^{-1}q^{-y} \\ & b_4, & q^{-N}, & b_2c^{-1} \end{matrix}; q, q\right]. $$

We now substitute these in (4.9), replace x, y by $N-x$ and $N-y$, respectively, and improve the notation somewhat by replacing c^{-1} by b_3. The final form of the formula is

(4.10)
$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n-1}b_2b_4, & q^{-x}, & b_1^{-1}b_2b_3q^{x-N} \\ & b_1^{-1}b_2b_4, & q^{-N}, & b_2b_3 \end{matrix}; q, q\right] $$
$$ \cdot{}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n-1}b_2b_4, & q^{-y}, & b_3q^{y-N} \\ & b_4, & q^{-N}, & b_2b_3 \end{matrix}; q, q\right] $$
$$ = \frac{(b_4b_3^{-1})_n(b_2b_4q^N)_n}{(b_2b_3)_n(q^{-N})_n}(b_4b_3^{-1}q^N)^{-n} $$
$$ \cdot\sum_{k=0}^{n}\frac{(q^{-n})_k(q^{n-1}b_2b_4)_k(b_4b_3^{-1}q^{N-x})_k(b_2b_1^{-1}b_4q^x)_k}{(q)_k(b_4)_k(b_2b_1^{-1}b_4)_k(b_4b_3^{-1})_k(b_4b_3^{-1}q^N)_k} $$

$$\cdot \frac{\left(b_4 b_3^{-1} q^{N-y}\right)_k \left(b_4 q^y\right)_n}{\left(b_2 b_4 q^N\right)_k} q^k$$

$$\cdot {}_{10}\Phi_9\left[\begin{matrix} b_3 b_4^{-1} q^{-N-k}, & q\sqrt{b_3 b_4^{-1} q^{-N-k}}, & q\sqrt{b_3 b_4^{-1} q^{-N-k}}, & b_2^{-1} b_4^{-1} q^{1-k-N}, \\ & \sqrt{b_3 b_4^{-1} q^{-N-k}}, & \sqrt{b_3 b_4^{-1} q^{-N-k}}, & b_2 b_3, \end{matrix}\right.$$

$$\begin{matrix} b_3 b_4^{-1} q^{1-k}, & q^{-k}, & b_1^{-1} b_2 b_3 q^{x-N}, & b_3 q^{y-N}, \\ q^{-N}, & b_3 b_4^{-1} q^{1-N}, & b_1 b_2^{-1} b_4^{-1} q^{1-x-k}, & b_4^{-1} q^{1-y-k}, \end{matrix}$$

$$\left.\begin{matrix} q^{-x}, & q^{-y} \\ b_3 b_4^{-1} q^{1-N+x-k}, & b_3 b_4^{-1} q^{1-N+y-k} \end{matrix} ; q, b_1 b_4^{-1} q\right],$$

where n is a nonnegative integer whose value is restricted by the requirement that none of the denominator parameters on the l.h.s. attains the value 1, if any of them do at all, before q^{-n} does.

5. Product formulas for q-Racah and q-Wilson polynomials. We would like to think of equation (4.10) as a master formula that leads to different types of product formulas for q-Racah polynomials defined in (1.2) as well as for q-Wilson polynomials **[10]** defined by

$$(5.1)\qquad p_n(x; a, b, c, d) = {}_4\Phi_3\left[\begin{matrix} q^{-n}, & q^{n-1}abcd, & ae^{i\theta}, & ae^{-i\theta} \\ & ab, & ac, & ad \end{matrix} ; q;, q\right],$$

where $x = \cos\theta$, $0 \leqslant \theta \leqslant \pi$.

For the q-Racah case let us specialize the parameters in the following way:

$$(5.2)\qquad \begin{gathered} b_2 = q^{M+\alpha+\beta+2}, \quad b_3 = q^{-(M+\beta+1)}, \quad b_4 = q^{-M}, \\ b_2 b_1^{-1} b_4 = q^{-M'}, \quad q^{-N} = q^{\beta+\gamma+1}, \end{gathered}$$

where M and M' are nonnegative integers. (4.10) then gives

$$\begin{aligned} &{}_4\Phi_3\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-x}, & q^{x-M'+\gamma} \\ & q^{\alpha+1}, & q^{-M'}, & q^{\beta+\gamma+1} \end{matrix} ; q, q\right] \\ &\qquad\cdot {}_4\Phi_3\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-y}, & q^{y-M+\gamma} \\ & q^{\alpha+1}, & q^{-M}, & q^{\beta+\gamma+1} \end{matrix} ; q, q\right] \end{aligned}$$

$$(5.3)\qquad = (q^{\beta+1})_n (q^{\alpha-\gamma+1})_n / (q^{\alpha+1})_n (q^{\beta+\gamma+1})_n$$

$$\cdot q^{\gamma n} \sum_{k=0}^{n} \frac{(q^{-n})_k (q^{n+\alpha+\beta+1})_k (q^{-\gamma-x})_k (q^{-\gamma-y})_k (q^{x-M'})_k (q^{y-M})_k}{(q)_k (q^{-M})_k (q^{-M'})_k (q^{\beta+1})_k (q^{-\gamma})_k (q^{\alpha-\gamma+1})_k}$$

$$\cdot q^k {}_{10}\Phi_9\left[\begin{matrix} q^{\gamma-k}, & q\sqrt{q^{\gamma-k}}, & -q\sqrt{q^{\gamma-k}}, & q^{\gamma-\alpha-k}, & q^{-\beta-k}, & q^{-k}, & q^{\gamma+x-M'}, \\ & \sqrt{q^{\gamma-k}}, & -\sqrt{q^{\gamma-k}}, & q^{\alpha+1}, & q^{\beta+\gamma+1}, & q^{\gamma+1}, & q^{M'-x+1-k}, \end{matrix}\right.$$

$$\left.\begin{matrix} q^{\gamma+y-M}, & q^{-x}, & q^{-y} \\ q^{M-y+1-k}, & q^{\gamma+x+1-k}, & q^{\gamma+y+1-k} \end{matrix} ; q, q^{M+M'+\alpha+\beta+3}\right].$$

Unless $M + M' + \alpha + \beta + 2 = 0$ the ${}_{10}\phi_9$ series on the right cannot be transformed to another ${}_{10}\phi_9$ which does not mean, however, that the sum on the r.h.s. cannot be transformed to any other form. All one needs to do is to use various choices of the parameters b_1, b_2, b_3, b_4, N so that the l.h.s. of (4.10) remains essentially the same while the r.h.s. undergoes a transformation. For example, an alternative to (5.2) is the choice

$$b_1 = q^{\alpha+\gamma+1}, \quad b_2 = q^{\alpha-\gamma+1}, \quad b_3 = q^{\gamma}, \quad b_4 = q^{\beta+\gamma+1} \tag{5.4}$$

which leads to

(5.5)

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-x}, & q^{x-N-\gamma} \\ & q^{\alpha+1}, & q^{-N}, & q^{\beta-\gamma+1} \end{matrix}; q, q\right] \\
&\qquad \cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-y}, & q^{y-N+\gamma} \\ & q^{\alpha+1}, & q^{-N}, & q^{\beta+\gamma+1} \end{matrix}; q, q\right] \\
&= \frac{(q^{\beta+1})_n (q^{N+\alpha+\beta+2})_n}{(q^{\alpha+1})_n (q^{-N})_n} q^{-(N+\beta+1)n} \\
&\quad \cdot \sum_{k=0}^{n} \frac{(q^{-n})_k (q^{n+\alpha+\beta+1})_k (q^{N+\beta+1-x})_k (q^{N+\beta+1-y})_k}{(q)_k (q^{\beta+1})_k (q^{N+\alpha+\beta+2})_k (q^{N+\beta+1})_k} \frac{(q^{\beta-\gamma+1+x})_k (q^{\beta+\gamma+1+y})_k}{(q^{\beta+\gamma+1})_k (q^{\beta-\gamma+1})_k} q^k \\
&\quad \cdot {}_{10}\phi_9\left[\begin{matrix} q^{-N-\beta-1-k}, & q\sqrt{q^{-N-\beta-1-k}}, & -q\sqrt{q^{-N-\beta-1-k}}, & q^{-N-\alpha-\beta-1-k}, & q^{-\beta-k}, \\ & \sqrt{q^{-N-\beta-1-k}}, & -\sqrt{q^{-N-\beta-1-k}}, & q^{\alpha+1}, & q^{-N}, \end{matrix}\right. \\
&\qquad\qquad \left.\begin{matrix} q^{-k}, & q^{x-N-\gamma}, & q^{\gamma-N+y}, & q^{-x}, & q^{-y} \\ q^{-\beta-N}, & q^{\gamma-\beta-x-k}, & q^{-\gamma-\beta-y-k}, & q^{x-N-\beta-k}, & q^{y-N-\beta-k} \end{matrix}; q, q^{\alpha-\beta+1}\right].
\end{aligned}$$

One can derive a number of things from (5.3) and (5.5). First of all we will show how (5.3) leads to a Watson-type product formula for the ${}_4\phi_3$ polynomials. Using (2.5) one can easily show that

$$\begin{aligned}
&\frac{(q^{-\gamma-x})_{k-l}(q^{-\gamma-y})_{k-l}(q^{x-M'})_{k-l}(q^{y-M})_{k-l}}{(q)_{k-l}(q^{\alpha-\gamma+1})_{k-l}(q^{-\gamma})_{k-l}(q^{\beta+1})_{k-l}} \\
&= \frac{(q^{-\gamma-x})_k(q^{-\gamma-y})_k(q^{x-M'})_k(q^{y-M})_k}{(q)_k(q^{\alpha-\gamma+1})_k(q^{-\gamma})_k(q^{\beta+1})_k} \\
&\quad \cdot \frac{(q^{\gamma-\alpha-k})_l(q^{\gamma+1-k})_l(q^{-\beta-k})_l(q^{-k})_l}{(q^{\gamma+x+1-k})_l(q^{\gamma+1+y-k})_l(q^{M'+1-x-k})_l(q^{M+1-y-k})_l} q^{(M+M'+\alpha+\beta+3)l}.
\end{aligned} \tag{5.6}$$

Using this in (5.3) and replacing the summation variables k, l by $r+s$ and r, respectively, we get

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-x}, & q^{x-M'+\gamma} \\ & q^{\alpha+1}, & q^{-M'}, & q^{\beta+\gamma+1} \end{matrix}; q, q\right] \\
&\quad \cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-y}, & q^{y-M+\gamma} \\ & q^{\alpha+1}, & q^{-M}, & q^{\beta+\gamma+1} \end{matrix}; q, q\right] \\
(5.7)\quad &= (q^{\beta+1})_n (q^{\alpha-\gamma+1})_n / (q^{\alpha+1})_n (q^{\beta+\gamma+1})_n \\
&\quad \cdot q^{\gamma n} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(q^{-n})_{r+s} (q^{n+\alpha+\beta+1})_{r+s} (q^{-x})_r (q^{-y})_r (q^{\gamma+x-M'})_r (q^{\gamma+y-M})_r}{(q)_r (q)_s (q^{-M})_{r+s} (q^{-M'})_{r+s} (q^{\alpha+1})_r (q^{\gamma+1})_r (q^{\beta+\gamma+1})_r} \\
&\quad \cdot \frac{(q^{x-M'})_s (q^{y-M})_s (q^{-\gamma-x})_s (q^{-\gamma-y})_s}{(q^{\beta+1})_s (q^{-\gamma}) (q^{\alpha-\gamma+1})_s} \frac{1-q^{\gamma-s+r}}{1-q^{\gamma-s}} q^{r+s}.
\end{aligned}
$$

This is an interesting formula which is likely to have important applications to convolution structures of all orthogonal ${}_4\phi_3$ polynomials. We hope to report on these applications in a forthcoming paper.

It is obvious that a similar formula can be obtained from (5.5), but its main advantage is revealed when we take the limit $\gamma \to \infty$ and obtain the formula for the q-Hahn polynomials:

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-x} \\ & q^{\alpha+1}, & q^{-N} \end{matrix}; q, q^{x-N-\beta}\right] {}_3\phi_2\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-y} \\ & q^{\alpha+1}, & q^{-N} \end{matrix}; q, q\right] \\
&\quad = (q^{\beta+1})_n (q^{N+\alpha+\beta+2})_n / (q^{\alpha+1})_n (q^{-N})_n \\
(5.8)\quad &\quad \cdot q^{-(N+\beta+1)n} \sum_{k=0}^{n} \frac{(q^{-n})_k (q^{n+\alpha+\beta+1})_k (q^{N+\beta+1-x})_k (q^{N+\beta+1-y})_k}{(q)_k (q^{\beta+1})_k (q^{N+\alpha+\beta+2})_k (q^{N+\beta+1})_k} \cdot q^{(x+1)k} \\
&\quad \cdot {}_8\phi_7\left[\begin{matrix} q^{-N-\beta-1-k}, & q\sqrt{q^{-N-\beta-1-k}}, & -q\sqrt{q^{-N-\beta-1-k}}, & q^{-N-\alpha-\beta-1-k}, & q^{-\beta-k}, \\ & \sqrt{q^{-N-\beta-1-k}}, & -\sqrt{q^{-N-\beta-1-k}}, & q^{\alpha+1}, & q^{-N}, \end{matrix}\right. \\
&\qquad\qquad \left.\begin{matrix} q^{-k}, & q^{-x}, & q^{-y} \\ q^{-N-\beta}, & q^{x-N-\beta-k}, & q^{y-N-\beta-k} \end{matrix}; q, q^{\alpha+1-N+x+y+k}\right].
\end{aligned}
$$

However, use of (2.19) now gives

$$
\begin{aligned}
{}_8\phi_7[\;] &= \frac{(q^{-N-\beta-k})_k (q^{x+y-N-\beta-k})_k}{(q^{x-N-\beta-k})_k (q^{y-N-\beta-k})_k} {}_4\phi_3\left[\begin{matrix} q^{-k}, & q^{k+\alpha+\beta+1}, & q^{-x}, & q^{-y} \\ & q^{\alpha+1}, & q^{-N}, & q^{N+\beta+1-x-y} \end{matrix}; q, q\right] \\
&= \frac{(q^{N+\beta+1})_k (q^{N+\beta+1-x-y})_k}{(q^{N+\beta+1-x})_k (q^{N+\beta+1-y})_k} {}_4\phi_3\left[\begin{matrix} q^{-k}, & q^{k+\alpha+\beta+1}, & q^{-x}, & q^{-y} \\ & q^{\alpha+1}, & q^{-N}, & q^{N+\beta+1-x-y} \end{matrix}; q, q\right]
\end{aligned}
$$

by (1.11). So equation (5.8) reduces to

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-x} \\ & q^{\alpha+1}, & q^{-N} \end{matrix}; q, q^{x-N-\beta}\right] {}_3\phi_2\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-y} \\ & q^{\alpha+1}, & q^{-N} \end{matrix}; q, q\right] \\
&\quad = (q^{\beta+1})_n (q^{N+\alpha+\beta+2})_n / (q^{\alpha+1})_n (q^{-N})_n \\
&\quad\quad \cdot q^{-(N+\beta+1)n} \sum_{k=0}^{n} \frac{(q^{-n})_k (q^{n+\alpha+\beta+1})_k (q^{N+\beta+1-x-y})_k}{(q_k)(q^{\beta+1})_k (q^{N+\alpha+\beta+2})_k} q^{(x+1)k} \\
&\quad\quad \cdot {}_4\phi_3\left[\begin{matrix} q^{-k}, & q^{k+\alpha+\beta+1}, & q^{-x}, & q^{-y} \\ & q^{\alpha+1}, & a^{-N}, & q^{N+\beta+1-x-y} \end{matrix}; q, q\right].
\end{aligned} \tag{5.9}
$$

If we let $q \to 1$ this becomes the Bateman-type product formula for the Hahn polynomials that was obtained in [**8**]. Unfortunately (5.9) cannot be strictly regarded as its proper q-analogue because of the argument $q^{x-N-\beta}$ in the first ${}_3\phi_2$ on the l.h.s. However, using (1.5) we may easily derive the following transformation formula:

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-x} \\ & q^{\alpha+1}, & q^{-N} \end{matrix}; q, q^{x-N-\beta}\right] \\
&\quad = (-1)^n \frac{(q^{\beta+1})_n}{(q^{\alpha+1})_n} q^{-n(n+1)/2-\beta n} {}_3\phi_2\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{x-N} \\ & q^{\beta+1}, & q^{-N} \end{matrix}; q, q\right].
\end{aligned} \tag{5.10}
$$

Substituting (5.10) in (5.9) and then replacing x by $N - x$ we obtain

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-x} \\ & q^{\beta+1}, & q^{-N} \end{matrix}; q, q\right] {}_3\phi_2\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-y} \\ & q^{\alpha+1}, & q^{-N} \end{matrix}; q, q\right] \\
&\quad = \frac{(q^{N+\alpha+\beta+2})_n}{(q^{-N})_n} (-1)^n q^{n(n-1)/2-Nn} \\
&\quad\quad \cdot \sum_{k=0}^{n} \frac{(q^{-n})_k (q^{n+\alpha+\beta+1})_k (q^{\beta+1+x-y})_k}{(q)_k (q^{\beta+1})_k (q^{N+\alpha+\beta+2})_k} q^{(N-x+1)k} \\
&\quad\quad \cdot {}_4\phi_3\left[\begin{matrix} q^{-k}, & q^{k+\alpha+\beta+1}, & q^{x-N}, & q^{-y} \\ & q^{\alpha+1}, & q^{-N}, & q^{\beta+1+x-y} \end{matrix}; q, q\right].
\end{aligned} \tag{5.11}
$$

The two q-Hahn polynomials on the l.h.s. have the same parameters only when $\alpha = \beta$, which seems to imply that a product formula of Bateman type does not exist for this particular q-analogue of the Hahn polynomials except in this special case. However, a Watson-type product formula for these polynomials always exists, as can be seen by taking the limit $\gamma \to \infty$ in (5.7):

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-x} \\ & q^{\alpha+1}, & q^{-M'} \end{matrix}; q, q\right] {}_3\phi_2\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{-y} \\ & q^{\alpha+1}, & q^{-M} \end{matrix}; q, q\right] \\
&\quad = (-1) q^{\alpha n+n(n+1)/2} \frac{(q^{\beta+1})_n}{(q^{\alpha+1})_n} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(q^{-n})_{r+s} (q^{n+\alpha+\beta+1})_{r+s} (q^{-x})_r (q^{-y})_r}{(q_r)(q)_s (q^{-M})_{r+s} (q^{-M'})_{r+s} (q^{\alpha+1})_r} \\
&\quad\quad \cdot \frac{(q^{x-M'})_s (q^{y-M})_s}{(q^{\beta+1})_s} q^{r-(\alpha+x+y)s}.
\end{aligned} \tag{5.12}
$$

This is a straight generalization of Gasper's product formula for Hahn polynomials **[6]**.

Let us now set $M = M'$ in (5.7) (although this specialization is not necessary) and introduce parameters a, b, c, d and real angles θ, ϕ through the correspondence:

$$\begin{aligned} q^{-x} = ae^{i\theta}, \quad q^{-y} = ae^{i\phi}, \quad q^{-M} = ad, \\ q^{\alpha+1} = ab, \quad q^{\beta+1} = cd, \quad q^{\gamma} = a/d. \end{aligned} \tag{5.13}$$

Since (5.7) remains valid no matter what values we choose for the parameters as long as n is a nonnegative integer, substitution of (5.13) in (5.7) now leads to an interesting formula for q-Wilson polynomials defined in (5.1):

$$\begin{aligned} p_n(x; a, b, c, d) p_n(y; a, b, c, d) = \frac{(cd)_n (bd)_n}{(ab)_n (ac)_n} \left(\frac{a}{d}\right)^n \\ \cdot \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(q^{-n})_{r+s} (q^{n-1}abcd)_{r+s} (ae^{-i\theta})_r (ae^{i\theta})_r (ae^{i\phi})_r (ae^{-i\phi})_r}{(q)_r (q)_s (ad)_{r+s} (ad)_{r+s} (ab)_r (aq/d)_r (ac)_r} \\ \cdot \frac{(de^{i\theta})_s (de^{-i\theta})_s (de^{i\phi})_s (de^{-i\phi})_s}{(cd)_s (bd)_s (d/a)_s} \cdot \frac{1 - aq^{r-s}/d}{1 - aq^{-s}/d} q^{r+s}. \end{aligned} \tag{5.14}$$

Note that the reality of the r.h.s. is self-evident as is its symmetry in θ, ϕ and b, c. The continuous q-Jacobi polynomials, as defined in **[10]**, are a special case of q-Wilson polynomials with $a = \sqrt{q} = -d$, $b = q^{\alpha+1/2}$, $c = -q^{\beta+1/2}$. We show the corresponding formula explicitly:

$$\begin{aligned} {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & \sqrt{q}\, e^{i\theta}, & \sqrt{q}\, e^{-i\theta} \\ & q^{\alpha+1}, & -q^{\beta+1}, & -q \end{matrix} ; q, q\right] \\ \cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & \sqrt{q}\, e^{i\phi}, & \sqrt{q}\, e^{-i\phi} \\ & q^{\alpha+1}, & -q^{\beta+1}, & -q \end{matrix} ; q, q\right] \\ = (q^{\beta+1})_n (-q^{\alpha+1})_n / (q^{\alpha+1})_n (-q^{\beta+1})_n \\ \cdot (-1)^n \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(q^{-n})_{r+s} (q^{n+\alpha+\beta+1})_{r+s} \left(\sqrt{q}\, e^{i\theta}\right)_r \left(\sqrt{q}\, e^{-i\theta}\right)_r}{(q)_r (q)_s (-q)_{r+s} (-q)_{r+s} (q^{\alpha+1})_r} \\ \cdot \frac{\left(\sqrt{q}\, e^{i\phi}\right)_r \left(\sqrt{q}\, e^{-i\phi}\right)_r \left(-\sqrt{q}\, e^{i\theta}\right)_s \left(-\sqrt{q}\, e^{-i\theta}\right)_s \left(-\sqrt{q}\, e^{i\phi}\right)_s \left(-\sqrt{q}\, e^{-i\phi}\right)_s}{(-q^{\beta+1})_r (-q)_r (q^{\beta+1})_s (-q^{\alpha+1})_s (-1)_s} \frac{1 + q^{r-s}}{1 + q^{-s}} q^{r+s}. \end{aligned} \tag{5.15}$$

In the ultraspherical case $\alpha = \beta$ one can derive a Bateman-type formula by setting $\alpha = \beta$ in (5.5), transforming the ${}_{10}\phi_9$ by a formula of type (3.14) so that the top left-hand parameter becomes free of k. But the resulting formula seems to be less interesting than, and strictly speaking, only a special case of equation (5.15) and so does not seem worth separate display.

REFERENCES

1. W. Al-Salam and M. E. H. Ismail, *Reproducing kernels for q-Jacobi polynomials*, Proc. Amer. Math. Soc. **67** (1977), 105–110.

2. G. E. Andrews and R. A. Askey, *Enumeration of partitions: the role of Eulerian series and q-orthogonal polynomials*, Higher Combinatorics (M. Ainger, ed.), Reidel, Dordrecht, 1977, pp. 3–26.

3. ______, *The classical and discrete orthogonal polynomials and their q-analogues* (in preparation).

4. R. A. Askey and J. Wilson, *A set of orthogonal polynomials that generalize the Racah coefficients or 6-j symbols*, SIAM J. Math. Anal. **10** (1979), 1008–1016.

5. R. A. Askey, *Orthogonal polynomials and special functions*, Regional Conf. Ser. Appl. Math., Vol. 21, SIAM, Philadelphia, 1975.

6. G. Gasper, *Non-negativity of a discrete Poisson kernel for the Hahn polynomials*, J. Math. Anal. Appl. **42** (1973), 438–451.

7. W. Hahn, *Über Orthogonalpolynome, die q-differenzengleichungen genugen*, Math. Nachr. **2** (1949), 4–34.

8. Mizan Rahman, *Some positive kernels and bilinear sums for Hahn polynomials*, SIAM J. Math. Anal. **7** (1976), 414–435.

9. ______, *A stochastic matrix and bilinear sums for Racah-Wilson polynomials*, SIAM J. Math. Anal. **12** (1981), 145–160.

10. ______, *The linearization of the product of continuous q-Jacobi polynomials*, Canad. J. Math. **33** (1981), 961–987.

11. L. J. Slater, *Generalized hypergeometric functions*, Cambridge Univ. Press, Cambridge, England, 1966.

DEPARTMENT OF MATHEMATICS AND STATISTICS, CARLETON UNIVERSITY, OTTAWA, ONTARIO, CANADA K1S 5B6